442 Tutorial Sheet 8 Solutions to Questions 2 and 3^1

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2 (4) After applying the harmonic gauge there is a residual invariance under coordinate transformation. For a gravitational wave in the $z$-direction, show that the polarization tensors can be taken to be $\epsilon_+\text{ and } \epsilon_-$ where $\epsilon_+ = \text{diag}(0,1,-1,0)$ and $(\epsilon_c)_{12} = (\epsilon_c)_{21} = 1$ and all others zero. This was sketched in class, the challenge here is to do it more completely.

Solution: So, we know that Harmonic gauge condition says

\[ k_c e^c_a - \frac{1}{2} k_a e^c_a = 0 \]  

and that the residual gauge transformations are of the form

\[ \epsilon_{ab}' = \epsilon_{ab} + k_a \alpha_b + k_b \alpha_a \]  

and for a $z$-polarized wave $k_a = (k, 0, 0, k)$. Putting this back into the gauge condition we get

\[ \epsilon_0^0 + \epsilon_3^3 - \frac{1}{2} \epsilon = 0 \]
\[ \epsilon_1^1 + \epsilon_1^1 = 0 \]
\[ \epsilon_2^2 + \epsilon_2^2 = 0 \]
\[ \epsilon_3^3 + \epsilon_3^3 - \frac{1}{2} \epsilon = 0 \]  

where $\epsilon = \epsilon_+$. Now taking the trace of the residual gauge

\[ \epsilon' = \epsilon + 2k^a \alpha_a = \epsilon + 2k(\alpha_3 - \alpha_0) \]  

and hence, by choice of $\alpha_3 - \alpha_0$ we can make $\epsilon = 0$. To preserve this we must have $\alpha_3 = \alpha_0$ in future transformations. Now, consider the zero-zero residual gauge

\[ \epsilon'_{00} = \epsilon_{00} + 2k_0 \alpha_0 = \epsilon_{00} + 2k_0 \alpha_0 \]  

which means that $\epsilon_{00}$ can be set to zero with a choice of $\alpha_0$. From the harmonic gauge condition this means $\epsilon_{00}$ is zero too, and using the symmetry of the polarization tensor, so is $\epsilon_{33}$. Next take the zero-one transformation

\[ \epsilon'_{01} = \epsilon_{01} + k_1 \alpha_1 + k_1 \alpha_0 = \epsilon_{01} + k_1 \alpha_1 \]  

and, again, by a choice of $\alpha_1$, we can make $\epsilon_{01}$ zero. From the harmonic gauge condition this also gets rid of $\epsilon_{31}$. $\epsilon_{32} = -\epsilon_{31}$ can be set to zero using $\alpha_2$, leaving only $\epsilon_{11}$, $\epsilon_{22}$ and $\epsilon_{12} = \epsilon_{21}$. Since we have made the polarization tensor traceless, this gives the answer.

3 (2) By changing coordinates, describe the effect of an $\epsilon_+$ polarized gravity wave.

So, by finding the eigenvectors of the submatrix

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

It is easy to check that the polarization tensor is diagonalized by the $\pi/4$ rotation

\[ x' = \frac{1}{\sqrt{2}}(x + y) \]
\[ y' = \frac{1}{\sqrt{2}}(x - y) \]  

From this we deduce that an $\epsilon_+$ polarized wave expands and contracts in the $x \pm y$-directions.

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