442 Tutorial Sheet 8 Solutions to Questions 2 and 3[1]

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2 (4) After applying the harmonic gauge there is a residual invariance under coördinate transformation. For a gravitational wave in the z-direction, show that the polarization tensors can be taken to be e_+ and $e_\times$ where $e_+ = \text{diag}(0,1,-1,0)$ and $(e_\times)_{12} = (e_\times)_{21} = 1$ and all others zero. This was sketched in class, the challenge here is to do it more completely.

Solution: So, we know that Harmonic gauge condition says

$$k_c e^c{}_a - \frac{1}{2} k_a e^c{}_c = 0 \tag{1}$$

and that the residual gauge transformations are of the form

$$e'_{ab} = e_{ab} + k_a \alpha_b + k_b \alpha_a \tag{2}$$

and for a z-polatized wave $k_a = (k, 0, 0, k)$. Putting this back into the gauge condition we get

$$\begin{aligned} e^0{}_0 + e^3{}_0 - \frac{1}{2} e &= 0 \\ e^0{}_1 + e^3{}_1 &= 0 \\ e^0{}_2 + e^3{}_2 &= 0 \\ e^0{}_3 + e^3{}_3 - \frac{1}{2} e &= 0 \end{aligned} \tag{3}$$

where $e = e^c{}_c$. Now taking the trace of the residual gauge

$$e' = e + 2k^a \alpha_a = e + 2k(\alpha_3 - \alpha_0) \tag{4}$$

and hence, by choice of $\alpha_3 - \alpha_0$ we can make $e = 0$. To preserve this we mush have $\alpha_3 = \alpha_0$ in future transformations. Now, consider the zero-zero residual gauge

$$e'_{00} = e_{00} + 2k_0 \alpha_0 = e_{00} + 2k\alpha_0 \tag{5}$$

which means that e_{00} can be set to zero with a choice of α_0. From the harmonic gauge condition this means e_{30} is zero too, and using the symmetry of the polarization tensor, so is e_{33}. Next take the zero-one transformation

$$e'_{01} = e_{01} + k_0 \alpha_1 + k_1 \alpha_0 = e_{01} + k\alpha_1 \tag{6}$$

and, again, by a choice of α_1 we can make e_{01} zero. From the hamonic gauge condition this also gets rid of e_{31}. $e_{02} = -e_{32}$ can be set to zero using α_2, leaving only e_{11}, e_{22} and $e_{12} = e_{21}$. Since we have made the polarization tensor traceless, this gives the answer.

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3 (2) By changing coördinates, describe the effect of an $e_\times$ polarized gravity wave.

So, by finding the eigenvectors of the submatrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

It is easy to check that the polarization tensor is diagonalized by the $\pi/4$ rotation

$$\begin{aligned} x' &= \frac{1}{\sqrt{2}}(x+y) \\ y' &= \frac{1}{\sqrt{2}}(x-y) \end{aligned} \tag{8}$$

From this we deduce that a $e_\times$ polarizated wave expands and contracts in the $x \pm y$-directions.