2 (4) After applying the harmonic gauge there is a residual invariance under coordinate transformation. For a gravitational wave in the $z$-direction, show that the polarization tensors can be taken to be $e_+$ and $e_\times$ where $e_+ = \text{diag}(0, 1, -1, 0)$ and $(e_\times)_{12} = (e_\times)_{21} = 1$ and all others zero. This was sketched in class, the challenge here is to do it more completely.

Solution: So, we know that Harmonic gauge condition says

$$k_c e^c_a - \frac{1}{2} k_a e^c_c = 0 \quad (1)$$

and that the residual gauge transformations are of the form

$$e'_{ab} = e_{ab} + k_a \alpha_b + k_b \alpha_a \quad (2)$$

and for a $z$-polarized wave $k_a = (k, 0, 0, k)$. Putting this back into the gauge condition we get

$$e'_0 + e'_3 - \frac{1}{2} e = 0$$
$$e'_1 + e'_2 = 0$$

where $e = e^c_c$. Now taking the trace of the residual gauge

$$e' = e + 2k^a \alpha_a = e + 2k(\alpha_3 - \alpha_0) \quad (4)$$

and hence, by choice of $\alpha_3 - \alpha_0$ we can make $e = 0$. To preserve this we must have $\alpha_3 = \alpha_0$ in future transformations. Now, consider the zero-zero residual gauge

$$e'_{00} = e_{00} + 2k_0 \alpha_0 = e_{00} + 2k \alpha_0 \quad (5)$$

which means that $e_{00}$ can be set to zero with a choice of $\alpha_0$. From the harmonic gauge condition this means $e_{30}$ is zero too, and using the symmetry of the polarization tensor, so is $e_{33}$. Next take the zero-one transformation

$$e'_{01} = e_{01} + k_0 \alpha_1 + k_1 \alpha_0 = e_{01} + k \alpha_1 \quad (6)$$

and, again, by a choice of $\alpha_1$ we can make $e_{01}$ zero. From the harmonic gauge condition this also gets rid of $e_{31}$. $e_{02} = -e_{32}$ can be set to zero using $\alpha_2$, leaving only $e_{11}$, $e_{22}$ and $e_{12} = e_{21}$. Since we have made the polarization tensor traceless, this gives the answer.

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1Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/442.html
By changing coördinates, describe the effect of an $e_x$ polarized gravity wave.

So, by finding the eigenvectors of the submatrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(7)

It is easy to check that the polarization tensor is diagonalized by the $\pi/4$ rotation

$$x' = \frac{1}{\sqrt{2}}(x + y)$$
$$y' = \frac{1}{\sqrt{2}}(x - y)$$

(8)

From this we deduce that a $e_x$ polarized wave expands and contracts in the $x \pm y$-directions.