442 Tutorial Sheet 7[1]

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1. (3) Find the Robinson-Walker solution with positive cosmological constant and no matter for all three values of k.

   Solution: So, from what we did in lectures, the accelleration and Freidmann equations for a matter-free Universe with non-zero Λ are

$$\begin{aligned} \frac{3\ddot{a}}{a} &= \Lambda \\ \frac{3\dot{a}^2}{a^2} + \frac{3k}{a^2} &= \Lambda \end{aligned} \tag{1}$$

   Now, the accelleration equation can be solved explicitely to give

$$a = A\cosh\kappa t + B\sinh\kappa t \tag{2}$$

   where $\kappa = \sqrt{\Lambda/3}$. All we have to do is substitute this into the Freidman equation and solve for A and B. In fact, there is some ambiguity, the Freidmann equation is only one equation, so lets start by imposing some initial conditions at $t = 0$. Obviously changing $t \to t = t_0$ is just a change of A and B. Now suppose we assume $t = 0$ corresponds to $a = 0$ as before, then $A = 0$ and the Freidman equation gives

$$\kappa^2\coth^2\kappa t + \frac{k}{A^2\sinh^2\kappa t} = \kappa^2 \tag{3}$$

   Next, using $\coth^2\kappa t - 1 = 1/\sinh^2\kappa t$, we see that $A = 1/\kappa$ is a solution provided that $k = -1$ and that there is no solution if $k = 1$ or $k = 0$. However, we must ask, can we choose $t = 0$ so that $a(0) = 0$ and the answer is that we can, provided there is a t_0 such that $a(t_0) = 0$, if there is then $t \to t - t_0$ gives the desired initial condition, however $a(t_0) = 0$ gives

$$0 = A + B\tanh\kappa t_0 \tag{4}$$

   and so this can only be the case if $|A| < |B|$.

   Alternatively we could apply the boundary condition

$$\frac{da}{dt} = 0 \tag{5}$$

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at $t = 0$. This would mean that $B = 0$ and we would end up with

$$\kappa^2 \tanh^2 \kappa t + \frac{k}{B^2 \cosh^2 \kappa t} = \kappa^2 \tag{6}$$

which is a solution with $B = 1/\kappa$ provided $k = 1$. However, the same sort of arguement as before shows that a solution of this sort only exists when $|B| < |A|$. Finally, for $|A| = |B|$ we have

$$a = Ce^{\pm\kappa t} \tag{7}$$

which is a solution provided $k = 0$.

In short, the accelleration equation is solved by the hyperbolic functions and the acceller-ation equation is used to find A and B, if $|A| < |B|$ we can take $A = 0$ by choice of t_0 and we get the $k = 1$ solution, if $|B| < |A|$ we can take $B = 0$ and get the $k = -1$ solution and if $|A| = |B|$ we get the $k = 0$ solutions. There are solutions for all three values of k.

2. (2) The *particle horizon* is the radius of the sphere of all particles that could be seen by us. It is the maximum straight line distance that could have been travelled by a light ray since the begining of the universe. Obviously, in a static universe this would be t_0. What is is for a $k = 0$ dust universe?

   Solution: So, if a photon is emitted it recedes for two reason; it is travelling through space and space is itself expanding. For photons $ds^2 = 0$ so, if a photon is emmitted in the past at the begining of the universe

   $$dt^2 = a^2 ds_{III}^2 \tag{8}$$

   and so the present distance to the photon measured in the fixed 3-space metric is

   $$s_{III} = \int_0^{t_0} \frac{dt}{a} \tag{9}$$

   where we have chose the positive root. The physical distance at fixed times is given by as_{III} so the particle hotizon is

   $$h_p = a_0 \int_0^{t_0} \frac{dt}{a} \tag{10}$$

   For a dust universe $a = Ct^{2/3}$ for some C and so

   $$h_p = 3t_0 \tag{11}$$

3. (2) What is the particle horizon for an inflating universe.

   Solution: For an inflating universe $a = Ce^{At}$ for positive constant A. Thus,

$$h_p = e^{At_0} \int \frac{dt}{e^{At}} \approx \frac{1}{A} e^{At_0} \tag{12}$$

   where we are now taking $t = 0$ to correspond to the start of inflation.

4. (1) Find an integral formala for the age of the universe with general k and $\Lambda \neq 0$. This integral is elliptic and can be integrated explicitely in terms of elliptic functions. This is not required here.

   Solution: So, following along like in the next question we have

$$H^2 = \Omega_M H^2 + \Omega_\Lambda H^2 - \frac{k}{a^2} \tag{13}$$

   and using the same substitutions as below we get

$$H^2 a^2 = \dot{a}^2 = \frac{B_M}{a} + B_\Lambda a^2 - k \tag{14}$$

   where B_M is what is called B_0 below and $B_\Lambda = H^2 \Omega_\Lambda$ which is a constant since $\Omega_\Lambda = \Lambda/(3H^2)$. Hence, the age of the Universe is given by the elliptic integral

$$t_0 = \int_0^a \frac{da}{\sqrt{B_\Lambda^2 + B_M/a - k}} \tag{15}$$

5. (3) Calculate the leading order correction to the age of a dust universe with $\Omega_0 = 1 + \epsilon$ and $\epsilon > 0$. We previously looked at $\Omega_0 = 1 - \epsilon$.

   Solution: So, the $k = 1$ this time and so the Freidmann equation is

$$H^2 = \Omega H^2 - \frac{1}{a^2} \tag{16}$$

   Evaluating at $t = t_0$ gives

$$a_0 = \frac{1}{H\sqrt{\Omega_0 - 1}} \tag{17}$$

   Next, for a dust universe

$$\rho = \frac{\rho_0 a_0^3}{a^3} \tag{18}$$

   or

$$H^2 \Omega = \frac{\Omega_0}{H_0(\Omega_0 - 1)^{3/2}} \frac{1}{a^3} \tag{19}$$

or, substituting back into the Freidmann equation and defining B_0 to simplify the constants

$$H^2 a^2 = B_0 \frac{1}{a} - 1 \tag{20}$$

and $H = \dot{a}a$ so this gives

$$t_0 = \int_0^{a_0} \frac{da}{\sqrt{B_0/a - 1}} \tag{21}$$

To do the integral, let $a = B_0 \sin^2\theta$ so $da = 2B_0 \cos\theta \sin\theta$ and hence

$$t_0 = 2B_0 \int_0^{\theta_0} \sin^2\theta d\theta = -B_0[\sin\theta\cos\theta - \theta]_0^{\theta_0} \tag{22}$$

where, of course,

$$\theta_0 = \sin^{-1}\sqrt{\frac{a_0}{B_0}} = \sin^{-1}\sqrt{\frac{\Omega_0 - 1}{\Omega_0}} \tag{23}$$

From this we can use Pythagorous to show

$$\cos\theta_0 = \sqrt{\frac{1}{\Omega_0}} \tag{24}$$

Putting all this together gives

$$t_0 = \frac{1}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}\left[\sin^{-1}\sqrt{\frac{\Omega_0 - 1}{\Omega_0}} - \frac{\sqrt{\Omega_0 - 1}}{\Omega_0}\right] \tag{25}$$

Now, let

$$\Omega_0 = 1 + \epsilon \tag{26}$$

with small ϵ and use the well know expansion for arcsin:

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + O(\epsilon^7) \tag{27}$$

This gives

$$\begin{aligned} t_0 &= \frac{1}{H_0}\frac{1+\epsilon}{\epsilon}\left[\frac{1}{\sqrt{1+\epsilon}} + \frac{1}{6}\frac{\epsilon}{(1+\epsilon)^{3/2}} + \frac{3}{40}\frac{\epsilon^2}{(1+\epsilon)^{5/2}} - \frac{1}{1+\epsilon}\right] \\ &= \frac{2}{3}\left(1 - \frac{1}{5}\epsilon\right) + O(\epsilon^2) \end{aligned} \tag{28}$$