

# 442 Tutorial Sheet 6 Solutions<sup>1</sup>

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1. (3) The equation of state is often written in *adiabatic form*

$$p = (\gamma - 1)\rho \quad (1)$$

where  $p$  is pressure and  $\rho$  is density and  $0 \leq \gamma \leq 2$  is the *adiabatic index* with  $\gamma = 0$  for dust and  $\gamma = 4/3$  for radiation. Calculate  $\rho(a)$  for general  $\gamma$ .  $k = 0$  calculate  $a(t)$ . Find the age of the universe for  $k = 0$  and general  $\gamma$ .

*Solution:* So, to calculate  $\rho(a)$  we use the fluid equation

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0 \quad (2)$$

which in this case reads

$$\dot{\rho} + \frac{3\gamma\dot{a}}{a}\rho = 0 \quad (3)$$

or

$$\frac{d}{dt}(a^{3\gamma}\rho) = 0 \quad (4)$$

and hence

$$\rho = \rho(t_0) \left[ \frac{a(t_0)}{a(t)} \right]^{3\gamma}. \quad (5)$$

To find  $a(t)$  we use the Friedmann equation, with  $k = 0$  and substituting for  $\rho$  this is

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi\rho(t_0)}{3} \left[ \frac{a(t_0)}{a(t)} \right]^{3\gamma} \quad (6)$$

this gives

$$a^{3\gamma/2-1}\dot{a} = \sqrt{\frac{8\pi}{3}\rho_0 a_0^{3\gamma}} \quad (7)$$

where we have chosen the positive root. Hence, integrating gives

$$a^{3\gamma/2} = \frac{3\gamma}{2} H_0 a_0^{3\gamma/2} t \quad (8)$$

where we have set the integration constant by requiring  $a(0) = 0$  and we have used

$$H_0 = \sqrt{\frac{8\pi\rho_0}{3}} \quad (9)$$

which follows from the definition of  $H$  and the Friedmann equation. Finally, putting  $t = t_0$  gives the age of the universe,

$$t_0 = \frac{2}{3\gamma} H_0^{-1} \quad (10)$$

2. (2) In the notation of the previous question, find  $\gamma$  so that the expansion rate is constant. With this value of  $\gamma$  find  $a(t)$  for  $k = 1$  and  $k = -1$ .

*Solution:* Well, from the previous question we have

$$\dot{a}^2 = \frac{8\pi}{3} \frac{\rho_0 a_0^{3\gamma}}{a^{3\gamma-2}} \quad (11)$$

so  $\dot{a}$  is constant if  $3\gamma = 2$ . With this value of  $\gamma$  the Friedmann equation for general  $k$  reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3} \frac{\rho_0 a_0^2}{a^2} \quad (12)$$

or

$$\dot{a}^2 + k = C \quad (13)$$

where

$$C = \frac{8\pi}{3} \rho_0 a_0^2 \quad (14)$$

and hence

$$a = \sqrt{C - kt} \quad (15)$$

3. (2) In the same notation, show

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega) \quad (16)$$

Define the *logarithmic scale factor*  $s = \log a$  and write an equation for  $d\Omega/ds$ . Notice that this formula gives a clear idea of how  $\Omega$  behaves.

*Solution:* From the definition of  $\Omega$  we have

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{8\pi\rho a^2}{3\dot{a}^2} \quad (17)$$

so, by differentiating

$$\dot{\Omega} = \frac{8\pi\rho a^2}{3\dot{a}^2} + \frac{16\pi\rho a}{3\dot{a}} - \frac{16\pi a^2 \ddot{a}}{3\dot{a}^2} \quad (18)$$

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Now, we replace  $\rho$  in terms of  $\Omega$ ,  $\dot{a}/a$  with  $H$  and use the acceleration equation to replace  $\ddot{a}$ :

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) = -\frac{4\pi}{3}\rho(3\gamma - 2) = -\frac{1}{2}\Omega H^2(3\gamma - 2) \quad (19)$$

and this gives

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega) \quad (20)$$

Now, if we change to  $s = \log a$  we have

$$\frac{d\Omega}{ds} = \dot{\Omega} \frac{dt}{ds} = \frac{\dot{\Omega}}{H} \quad (21)$$

and so

$$\frac{d\Omega}{ds} = (2 - 3\gamma)\Omega(1 - \Omega) \quad (22)$$

We can see from this that if  $\gamma > 2/3$  then  $d\Omega/ds$  is positive for  $\Omega > 1$  and negative for  $\Omega < 1$  and so  $|1 - \Omega|$  always gets bigger. For  $\gamma < 2/3$  the opposite is true.