442 Tutorial Sheet 6 Solutions[1]

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1. (3) The equation of state is often written in *adiabatic form*

$$p = (\gamma - 1)\rho \tag{1}$$

where p is pressure and ρ is density and $0 \leq \gamma \leq 2$ is the *adiabatic index* with $\gamma = 0$ for dust and $\gamma = 4/3$ for radiation. Calculate $\rho(a)$ for general γ. $k = 0$ calculate $a(t)$. Find the age of the universe for $k = 0$ and general γ.

Solution: So, to calculate $\rho(a)$ we use the fluid equation

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0 \tag{2}$$

which in this case reads

$$\dot{\rho} + \frac{3\gamma\dot{a}}{a}\rho = 0 \tag{3}$$

or

$$\frac{d}{dt}\left(a^{3\gamma}\rho\right) = 0 \tag{4}$$

and hence

$$\rho = \rho(t_0)\left[\frac{a(t_0)}{a(t)}\right]^{3\gamma}. \tag{5}$$

To find $a(t)$ we use the Freidmann equation, with $k = 0$ and substituting for ρ this is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho(t_0)}{3}\left[\frac{a(t_0)}{a(t)}\right]^{3\gamma} \tag{6}$$

this gives

$$a^{3\gamma/2-1}\dot{a} = \sqrt{\frac{8\pi}{3}\rho_0 a_0^{3\gamma}} \tag{7}$$

where we have chosen the positive root. Hence, integrating gives

$$a^{3\gamma/2} = \frac{3\gamma}{2}H_0 a_0^{3\gamma/2} t \tag{8}$$

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where we have set the integration constant by requiring $a(0) = 0$ and we have used

$$H_0 = \sqrt{\frac{8\pi\rho_0}{3}} \tag{9}$$

which follows from the definition of H and the Freidman equation. Finally, putting $t = t_0$ gives the age of the universe,

$$t_0 = \frac{2}{3\gamma} H_0^{-1} \tag{10}$$

2. (2) In the notation of the previous question, find γ so that the expansion rate is constant. With this value of γ find $a(t)$ for $k = 1$ and $k = -1$.

   Solution: Well, from the previous question we have

$$\dot{a}^2 = \frac{8\pi}{3} \frac{\rho_0 a_0^{3\gamma}}{a^{3\gamma-2}} \tag{11}$$

   so $\dot{a}$ is constant if $3\gamma = 2$. With this value of γ the Freidmann equation for general k reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3} \frac{\rho_0 a_0^2}{a^2} \tag{12}$$

   or

$$\dot{a}^2 + k = C \tag{13}$$

   where

$$C = \frac{8\pi}{3} \rho_0 a_0^2 \tag{14}$$

   and hence

$$a = \sqrt{C - k} t \tag{15}$$

3. (2) In the same notation, show

$$\dot{\Omega} = (2 - 3\gamma) H \Omega (1 - \Omega) \tag{16}$$

   Define the *logarithmic scale factor* $s = \log a$ and write an equation for $d\Omega/ds$. Notice that this formula gives a clear idea of how Ω behaves.

   Solution: From the definition of Ω we have

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{8\pi\rho a^2}{3\dot{a}^2} \tag{17}$$

   so, by differenciating

$$\dot{\Omega} = \frac{8\pi\dot{\rho}a^2}{3\dot{a}^2} + \frac{16\pi\rho a}{3\dot{a}} - \frac{16\pi a^2 \ddot{a}}{3\dot{a}^2} \tag{18}$$

Now, we replace ρ in terms of Ω, $\dot{a}/a$ with H and use the accelleration equation to replace $\ddot{a}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) = -\frac{4\pi}{3}\rho(3\gamma - 2) = -\frac{1}{2}\Omega H^2(3\gamma - 2) \tag{19}$$

and this gives

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega) \tag{20}$$

Now, if we change to $s = \log a$ we have

$$\frac{d\Omega}{ds} = \dot{\Omega}\frac{dt}{ds} = \frac{\dot{\Omega}}{H} \tag{21}$$

and so

$$\frac{d\Omega}{ds} = (2 - 3\gamma)\Omega(1 - \Omega) \tag{22}$$

We can see from this that if $\gamma > 2/3$ then $d\Omega/ds$ is positive for $\Omega > 1$ and negative for $\Omega < 1$ and so $|1 - \Omega|$ always gets bigger. For $\gamma < 2/3$ the opposate is true.