1. (2) This question is revision of Lagrangian mechanics. By varying the field derive the Euler-Lagrange equations from $\delta S = 0$ where

$$S = \int L(t, q, \dot{q}) dt$$

Solution: Well vary $q$ to get

$$\delta L(t, q, \dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_i \frac{\partial L}{\partial q_i} \delta q_i$$

or

$$\delta S = \int \left( \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_i \frac{\partial L}{\partial q_i} \delta q_i \right) dt$$

and then use integration by part on the second term, assuming that there are fixed boundary conditions there is no boundary term and

$$\delta S = \int \left( \sum_i \frac{\partial L}{\partial \dot{q}_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial q_i} \right) \delta \dot{q}_i dt$$

and if this is true for all variations satisfying suitable boundary conditions, we get the Euler-Lagrange equation

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial q_i} = 0$$

2. (2) Derive the Euler-Lagrange equation

$$\frac{\partial}{\partial \phi} \frac{\partial L}{\partial (\partial_\phi \phi)} - \frac{\partial L}{\partial \phi} = 0$$

from $\delta S = 0$ where

$$S = \int L(t, \phi, \partial_\phi \phi) d^4x$$

Solution: This one isn’t really so different, bearing in mind $\partial_\mu \phi = \partial_\phi \phi$ nothing changes until we use the Gauss law, then we get

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{-g} - \frac{d}{dx} \frac{\partial L}{\partial (\partial_\phi \phi)} \right) \delta \phi d^4x$$

and then commute the $\sqrt{-g}$ through to give

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{-g} - \frac{\partial L}{\partial (\partial_\phi \phi)} \right) \delta \phi d^4x$$

and rewrite the $\partial_\phi \phi$ as $\partial_\mu \phi$ to get the answer.

3. (3) Derive the Euler-Lagrange equation

$$\nabla_\mu \frac{\partial L}{\partial (\nabla_\mu \phi)} - \frac{\partial L}{\partial \phi} = 0$$

from $\delta S = 0$ where

$$S = \int L(t, \phi, \nabla_\phi \phi) \sqrt{-g} d^4x$$

Recall that $\nabla_\mu \phi = \partial_\mu \phi$ and that $\sqrt{-g} \nabla_\mu X^\mu = \partial_\mu \sqrt{-g} X^\mu$.

Solution: This one isn’t really so different, bearing in mind $\nabla_\mu \phi = \partial_\phi \phi$ nothing changes until we use the Gauss law, then we get

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{-g} - \frac{d}{dx} \frac{\partial L}{\partial (\partial_\phi \phi)} \right) \delta \phi d^4x$$

and then commute the $\sqrt{-g}$ through to give

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{-g} - \frac{\partial L}{\partial (\partial_\phi \phi)} \right) \delta \phi d^4x$$

and rewrite the $\partial_\phi \phi$ as $\nabla_\mu \phi$ to get the answer.

4. (4) Calculate the energy-momentum tensor

$$T_{\mu\nu} = F_{\lambda\mu} F^\lambda_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}$$

for the Maxwell field:

$$S = \frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$
Solution: So, we need to write the action in a way that exposes the dependence on the metric,

$$S = -\frac{1}{4} \int g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \sqrt{g} d^4x$$

and then use

$$\delta g_{\mu\nu} = h_{\mu\nu}$$
$$\delta g^{\mu\nu} = -h^{\mu\nu}$$

$$\delta \sqrt{g} = \frac{1}{2} \sqrt{g} h$$

where, as usual, we are using the notation $h = g^{\mu\nu} h_{\mu\nu}$. Now,

$$S = \int \mathcal{L} d^4x$$

so that

$$\delta S = - \int \left( \frac{1}{2} h^{\mu\nu} g^{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} + \frac{1}{8} g^{\mu\nu} g^{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} h \right) \sqrt{g} d^4x$$

$$= \int \left( \frac{1}{2} F_{\rho\sigma} F_{\mu\nu}^{\rho\sigma} - \frac{1}{8} F_{\rho\sigma} F_{\mu\nu} g_{\rho\sigma} \right) h^{\mu\nu} \sqrt{g} d^4x$$

hence

$$T_{\lambda\mu} = 2 \frac{\delta S}{\delta h_{\lambda\mu}}$$
$$= F_{\lambda\rho} F_{\mu}^{\rho} - \frac{1}{4} F_{\rho\sigma} F_{\mu\nu} g_{\rho\sigma}$$

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