1. (2) This question is revision of Lagrangian mechanics. By varying the field derive the Euler-Lagrange equations from \( \delta S = 0 \) where

\[
S = \int L(t, q, \dot{q}) dt
\]  

(1)

\textbf{Solution:} Well vary \( q \) to get

\[
\delta L(t, q, \dot{q}) = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i
\]

(2)

or

\[
\delta S = \int \left( \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt
\]

(3)

and then use integration by part on the second term, assuming that there are fixed boundary conditions there is no boundary term and

\[
\delta S = \int \left( \sum_i \frac{\partial L}{\partial q_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt
\]

(4)

and if this is true for all variations satisfying suitable boundary conditions, we get the Euler-Lagrange equation

\[
\sum_i \frac{\partial L}{\partial q_i} - \sum_i \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0
\]

(5)

2. (2) Derive the Euler-Lagrange equation

\[
\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial (\partial \phi)} \frac{\partial L}{\partial \phi} = 0
\]

(6)

from \( \delta S = 0 \) where

\[
S = \int L(t, \phi, \partial_\mu \phi) d^4x
\]

(7)

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Solution: Well this is almost the same, vary $\phi$ and we get, by Taylor expanding $L$

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi \right) d^4 x \tag{8}$$

and then use a Gauss law and zero boundary conditions to give

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\phi d^4 x \tag{9}$$

and arbitrariness of $\delta \phi$ gives the Euler-Lagrange equations.

3. (3) Derive the Euler-Lagrange equation

$$\nabla_\mu \frac{\partial L}{\partial (\nabla_\mu \phi)} - \frac{\partial L}{\partial \phi} = 0 \tag{10}$$

from $\delta S = 0$ where

$$S = \int L(t, \phi, \nabla_\mu \phi) \sqrt{g} d^4 x \tag{11}$$

Recall that $\nabla_\mu \phi = \partial_\mu \phi$ and that $\sqrt{g} \nabla_\mu X^\mu = \partial_\mu \sqrt{g} X^\mu$.

Solution: This one isn’t really so different, bearing in mind $\nabla_\mu \phi = \partial_\mu \phi$ nothing changes until we use the Gauss law, then we get

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{g} - \frac{\partial}{\partial x^\mu} \sqrt{g} \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\phi d^4 x \tag{12}$$

and then commute the $\sqrt{g}$ through to give

$$\delta S = \int \left( \frac{\partial L}{\partial \phi} \sqrt{g} - \sqrt{g} \nabla_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\phi d^4 x \tag{13}$$

and rewrite the $\partial_\mu$s as $\nabla_\mu$ to get the answer.

4. (4) Calculate the energy-momentum tensor

$$T_{\mu\nu} = F_{\lambda\mu} F^{\lambda\nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \tag{14}$$

for the Maxwell field:

$$S = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \sqrt{g} d^4 x \tag{15}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{16}$$
Solution: So, we need to write the action in a way that exposes the dependence on the metric,

\[
S = -\frac{1}{4} \int g^{\lambda \mu} g^{\rho \sigma} F_{\lambda \rho} F_{\mu \sigma} \sqrt{g} d^4 x
\]  

(17)

and then use

\[
\frac{\delta g_{\mu \nu}}{\delta g^{\mu \nu}} = h_{\mu \nu}, \quad \frac{\delta \sqrt{g}}{\delta g^{\mu \nu}} = -\frac{1}{2} h_{\mu \nu}, \quad \frac{\delta \sqrt{g}}{\delta \sqrt{g}} = \frac{1}{2} \sqrt{g} h
\]  

(18)

where, as usual, we are using the notation \( h = g^{\mu \nu} h_{\mu \nu} \). Now,

\[
S = \int L d^4 x
\]  

(19)

so that

\[
\delta S = -\int \left( -\frac{1}{2} h^{\lambda \mu} g^{\rho \sigma} F_{\lambda \rho} F_{\mu \sigma} + \frac{1}{8} g^{\lambda \mu} g^{\rho \sigma} F_{\lambda \rho} F_{\mu \sigma} h \right) \sqrt{g} d^4 x
\]

(20)

\[
= \int \left( \frac{1}{2} F_{\lambda \rho} F_{\mu}^{\rho} - \frac{1}{8} F^{\rho \sigma} F_{\rho \sigma} g_{\lambda \mu} \right) h^{\lambda \mu} \sqrt{g} d^4 x
\]

(21)

hence

\[
T_{\lambda \mu} = 2 \frac{\delta S}{\delta h^{\lambda \mu}} = F_{\lambda \rho} F_{\mu}^{\rho} - \frac{1}{4} F^{\rho \sigma} F_{\rho \sigma} g_{\lambda \mu}
\]

(22)