442 Tutorial Sheet 4 Solutions[1]

1. (4) Find the Killing vectors for flat space $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$. [Write out Killing's equation in flat space, differenciate it once and then solve the resulting differential equation].

   Solution: The Killing's equation in flat space is

   $$k_{a,b} + k_{b,a} = 0 \tag{1}$$

   Clearly three solutions are given by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Also, if $a=b$ we have $k_{a,a} = 0$ where there is no sum on the a. Differenciating the above equation we find that

   $$k^{;a}_{a,b} + k^{;a}_{b,a} = k^{;a}_{b,a} = 0 \tag{2}$$

   so the k_b are all harmonic functions, ignoring the constant part, already mentioned, this gives $k = (a_1 y + a_2 z, a_3 x + a_4 z, a_5 x + a_6 y)$. Now, substitute this back into the Killing's equations, remembering that the three $a=b$ equations have already been solve,

   $$\begin{aligned} a_1 + a_3 &= 0 \\ a_2 + a_5 &= 0 \\ a_4 + a_6 &= 0 \end{aligned} \tag{3}$$

   Thus, a basis is given by $(y, -x, 0)$, $(-z, 0, x)$ and $(0, z, -y)$.

2. (3) This question concerns the weak field limit. Let

   $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

   where, by a choice of coordinates, $h_{\mu\nu}$ satisfies the harmonic gauge condition:

   $$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h \tag{5}$$

   show that the Einstein equations reduce to

   $$-\frac{1}{2}\partial^2 h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\partial^2 h = 8\pi T_{\mu\nu} \tag{6}$$

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where, as usual, $h = h^\mu_\mu$ is the trace. By taking the trace of both sides and solving for $\partial^2 h$, show that this can be written as

$$\partial^2 h_{\mu\nu} = -16\pi\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{7}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$. These are the linearized Einstein equations.

Solution: So the idea here is to work things out to first order in $h_{\mu\nu}$ and then simplify using the harmonic gauge. Rememeber that to leading order the indices on $h_{\mu\nu}$ are raised and lowered using $\eta_{\mu\nu}$. To leading order

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &\approx \frac{1}{2}\eta^{\lambda\rho}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right) \\ &= \frac{1}{2}\left(\partial_\mu h^\lambda_\nu + \partial_\nu h^\lambda_\mu - \partial^\lambda h_{\mu\nu}\right) \end{aligned} \tag{8}$$

The Riemann tensor is

$$R_{\mu\nu\lambda}{}^\rho \approx \partial_\nu \Gamma^\rho_{\mu\lambda} - \partial_\mu \Gamma^\rho_{\nu\lambda} \tag{9}$$

and in fact we only need the Ricci tensor

$$R_{\mu\lambda} = R_{\mu\rho\lambda}{}^\rho \approx \partial_\rho \Gamma^\rho_{\mu\lambda} - \partial_\mu \Gamma^\rho_{\rho\lambda} \tag{10}$$

Substituting from above, this gives

$$2R_{\mu\lambda} = \partial_\mu\partial_\rho h^\rho_\lambda + \partial_\lambda\partial_\rho h^\rho_\mu - \partial^2 h_{\mu\lambda} - \partial_\mu\partial_\lambda h \tag{11}$$

and

$$2R = 2\partial^\lambda\partial_\rho h^\rho_\lambda - 2\partial^2 h \tag{12}$$

Putting this together to form the Einstein tensor $2G_{\mu\nu} = 2R_{\mu\nu} - g_{\mu\nu}R$ and again replacing $g_{\mu\nu}$ with $\eta_{\mu\nu}$ to leading order

$$2G_{\mu\lambda} = \partial_\mu\partial_\rho h^\rho_\lambda + \partial_\lambda\partial_\rho h^\rho_\mu - \partial^2 h_{\mu\lambda} - \partial_\mu\partial_\lambda h - \eta_{\mu\lambda}\partial^\nu\partial_\rho h^\rho_\nu + \eta_{\mu\lambda}\partial^2 h \tag{13}$$

Now, recall the harmonic gauge condition:

$$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h \tag{14}$$

Differenciating this and changing indices

$$\partial_\lambda\partial_\nu h^\nu_\mu = \frac{1}{2}\partial_\mu\partial_\lambda h \tag{15}$$

There is a similar equation with μ and λ interchanged and taken together the first two terms in $G_{\mu\lambda}$ cancel with the fourth term. Differenciating the harmonic relation with respect to ∂_ν gives

$$\partial^\nu \partial_\mu h^\mu_\nu = \frac{1}{2}\partial^2 h \tag{16}$$

and so the second last term cancels half the last term. Putting all this together gives

$$-\frac{1}{2}\partial^2 h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\partial^2 h = 8\pi T_{\mu\nu} \tag{17}$$

as required.

Multiplying both sides by $\eta^{\mu\nu}$ we get

$$-\frac{1}{2}\partial^2 h + \partial^2 h = 8\pi T \tag{18}$$

or

$$\partial^2 h = 16\pi T \tag{19}$$

and substituting this back in gives the linearized Einstein equations.

3. (4) Cosmic strings are long heavy string-like objects which are possibly formed in the Universe during certain cooling transitions. They were once thought to be resposible for structure formation. In Cartesian coordinates, the energy momentum tensor for a cosmic string aligned along the z-axis may be approximated by

$$T_{\mu\nu} = \mu\delta(x)\delta(y)\text{diag}(1,0,0,-1) \tag{20}$$

where μ is a small positive constant. Working to linear order in μ show, using the linearized equations above that the change to the Minkowski metric is given by

$$h_{11} = h_{22} = -8\mu\log\frac{r}{r_0} \tag{21}$$

with all others perturbations zero. Here, $r = \sqrt{x^2+y^2}$ and r_0 is a length-scale which cannot be determined here, physically it is related to the radius of validity of the thin string approximation. By a change of variable in r

$$\left(1 - 8\mu\log\frac{r}{r_0}\right) r^2 = (1-8\mu)r'^2 \tag{22}$$

and in the azimuthal angle ϕ, show that the metric can still be written in the cylindrical form

$$ds^2 = -dt^2 + dz^2 + dr'^2 + r'^2 d\phi'^2 \tag{23}$$

but, the new azimuthal angle ϕ' does not have period 2π.

Solution: First we apply the weak field equations. First $T = \eta^{\mu\nu}T_{\mu\nu} = -2\mu\delta(x)\delta(y)$ and hence

$$\partial^2 h_{\mu\nu} = -16\pi\mu\delta(x)\delta(y)\text{diag}\,(0,1,1,0) \tag{24}$$

So the weak field equations are

$$\begin{aligned} \partial^2 h_{11} &= -16\pi\mu\delta(x)\delta(y) \\ \partial^2 h_{22} &= -16\pi\mu\delta(x)\delta(y) \end{aligned} \tag{25}$$

and $\partial^2 h_{\mu\nu} = 0$ otherwise. There is a bit of difficulty in applying the boundary conditions, but the arguement goes something like, if possible $h_{\mu\nu}$ should go to zero far from the cosmic string and, since the cosmic string is invariant under z-translations, so should the solutions. Hence $h_{\mu\nu} = 0$ for all components except h_{11} and h_{22}, these satisfy the two-dimensional Laplace equation everywhere except at $x = y = 0$.

Now, the solution to the two-dimensional Laplace equation is a log, let

$$h_{11} = A\log\frac{r}{r_0} \tag{26}$$

then

$$\frac{\partial}{\partial x}h_{11} = \frac{Ax}{r^2} \tag{27}$$

and

$$\frac{\partial^2}{\partial x^2}h_{11} = \frac{A}{r^2} - \frac{2Ax^2}{r^4} \tag{28}$$

Hence,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h_{11} = 0 \tag{29}$$

everywhere except at $r = 0$.

Next, we have to check that that

$$\int_D \partial^2 h_{11}dxdy = -\int_D 16\pi\mu\delta(x)\delta(y)dxdy = -16\pi\mu \tag{30}$$

where D is a disc of radius ρ around $x = y = 0$. The left hand side integral can be performed using Green's theorem on the plane:

$$\int_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy = \oint_C (f_1 dx + f_2 dy) \tag{31}$$

The integral we are interested in is of this form, with

$$
\begin{aligned}
f_1 &= -\frac{\partial h_{11}}{\partial y} = -\frac{Ay}{r^2} \\
f_2 &= \frac{\partial h_{11}}{\partial x} = \frac{Ax}{r^2}
\end{aligned} \tag{32}
$$

so that

$$\int_D \partial^2 h_{11} dx dy = -\oint_C \left(\frac{Aydx}{r^2} - \frac{Axdy}{r^2} \right) \tag{33}$$

where C is a circle of radius ρ, on the circle, $r = \rho$, $x = \rho \cos\phi$ and $y = \rho \sin\phi$, substituting this back in we get

$$\int_D \partial^2 h_{11} dx dy = A \oint d\phi = 2\pi A r_0 \tag{34}$$

and hence

$$A = -8\mu \tag{35}$$

as required.

Now, if we use polar coordinates for x and y we have

$$dx^2 + dy^2 = dr^2 + r^2 d\phi^2 \tag{36}$$

and the approximate cosmic string metric is

$$ds^2 = -dt^2 + dz^2 + \left(1 - 8\mu \log\frac{r}{r_0}\right)(dr^2 + r^2 d\phi^2) \tag{37}$$

Now consider the change of variables,

$$\left(1 - 8\mu \log\frac{r}{r_0}\right) r^2 = (1 - 8\mu) r'^2 \tag{38}$$

this gives

$$-\frac{8\mu}{r} r^2 dr + 2\left(1 - 8\mu \log\frac{r}{r_0}\right) r dr = 2(1 - 8\mu) r' dr' \tag{39}$$

or

$$\left(1 - 4\mu - 8\mu \log\frac{r}{r_0}\right) r dr = (1 - 8\mu) r' dr' \tag{40}$$

and substituting for the r factor on the left hand side, this gives

$$\left(1 - 4\mu - 8\mu \log\frac{r}{r_0}\right)\left[\left(1 - 4\mu - 8\mu \log\frac{r}{r_0}\right)(1 - 8\mu)\right]^{-1/2} dr = dr' \tag{41}$$

and then Taylor expand the left hand side to leading order in μ this gives

$$\left(1 - 4\mu \log\frac{r}{r_0}\right) dr = dr' \tag{42}$$

or, again, by Taylor expansion,

$$\left(1 - 8\mu \log\frac{r}{r_0}\right) dr^2 = dr'^2 \tag{43}$$

If we use all of this in our metric, we get

$$ds^2 = -dt^2 + dz^2 + dr'^2 + (1 - 8\mu) r'^2 d\phi^2 \tag{44}$$

Finally, let $\phi = (1 - 4\mu)\phi$ and we have

$$ds^2 = -dt^2 + dz^2 + dr'^2 + r'^2 d\phi'^2 \tag{45}$$

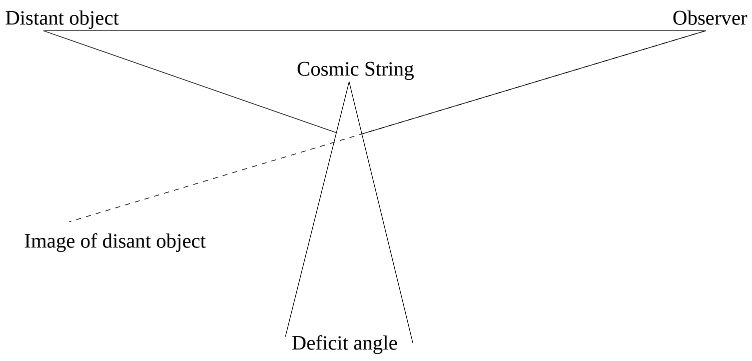
and the point is that this is just like ordinary flat space except that ϕ' has a different range: $0 \leq \phi' \leq (1 - 4\mu)2\pi$. If is like a wedge of angle $8\pi\mu$ has been cut out of space. This is called a conical singularity. As we will argue in the next question it causes image doubling. This calculation first appeared in

A. Vilenkin, *Gravitational field of vacuum domain walls and strings*, Phys. Rev. D **23** (1981) 852. and can be downloaded from the PRD website `http://prd.aps.org`.

4. (2) We have seen in the last question that a cosmic string causes a conical singularity, space is flat, but the azimuthal angle has period less than 2π. Argue that cosmic strings cause double images of distant objects.

Solution: So we have seen the space-time far from a cosmic string is like flat space but with a deficit angle $8\pi\mu$; the azimuthal angle doesn't have the full range: $0 \leq \phi' \leq 2\pi - 8\pi\mu$. This space time can be modelled as ordinary space-time but with a wedge missing of angle $8\pi\mu$. Opposate points on each side of the wedge are identified. Another version of this model is to join across the wedge to give a cone. There is a curvature singularity at the point, in a real cosmic string this singularity would be resolved by the full non-linear behaviour.

The figure show how this results in double images. There are two light rays that reach the observer, one travels straight from the distant object to the eye, the other crosses the wedge. Crossing the wedge deflect the ray by $4\mu\pi$ and so exactly one other ray from the distant object will reach the eye, causing a second image.


Distant object
Observer
Cosmic String
Image of disant object
Deficit angle