442 Tutorial Sheet 3 Solutions[1]

1. (3) What is the curvature on a cylinder?

   Solution: So the best thing is to go to cylindral coördinates,

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \tag{1}$$

   lies on a cylinder of radius r for all values of z and ϕ. Work our dx, dy and dz, Fix r so $dr = 0$ and substitute back into $ds^2 = dx^2 + dy^2 + dz^2$ to get

$$ds^2 = r^2 d\phi^2 + dz^2 \tag{2}$$

   and, since r is a constant, the metric coefficients are all constants and hence the connection coefficients are all zero.

2. (4) The Weyl tensor is described as being the trace-free part of the Riemann tensor. It is a (0,4) tensor with the same symmetries as the Riemann tensor, linear in the Riemann tensor, having no dependence on derivatives of the metric except through the Riemann tenson and with all traces vanishing. Find a formula for the Weyl tensor in terms of the Riemann tensor, the Ricci tensor and the Ricci scalar. [The easiest way to do this is to write the most general expression with the correct symmetries and the determine any arbitrary constants by contracting and using the defining quality of the Weyl tensor: it has zero traces].

   Solution: This is done by writing down a general expression with the right symmetries and then imposing the tracefree conditions which define the Weyl tensor. Now, the Weyl tensor has the same symmetries as the Riemann tensor and depends only on the metric and linearly on the Riemann tensor, the Ricci tensor and the Ricci scalar. The only possible Ricci terms are of the form $R_{ab}g_{cd}$ and $Rg_{ab}g_{cd}$ so applying the symmetries tell us that

$$W_{abcd} = R_{abcd} + A(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) + BR(g_{ac}g_{bd} - g_{bc}g_{ad}) \tag{3}$$

   Now, we know that

$$g^{ac}W_{abcd} = 0 \tag{4}$$

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so

$$R_{bd} + ARg_{bd} + 4AR_{bd} - AR_{bd} - AR_{bd} + 4BRg_{bd} - BRg_{bd} = 0 \tag{5}$$

or

$$R_{bd} + ARg_{bd} + 2AR_{bd} + 3BRg_{bd} = 0 \tag{6}$$

and hence $A = -1/2$ and $B = 1/6$.

3. (3) Find the time-like geodesics for the metric

$$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right) \tag{7}$$

You might want to use the integral

$$\int \frac{dt}{t\sqrt{1+C^2t^2}} = \frac{1}{2}\log\left(\frac{\sqrt{1+C^2t^2}-1}{\sqrt{1+C^2t^2}+1}\right) \tag{8}$$

[Rather than writting out the geodesic equations, it may be easier to note that x is ignorable and then use proper time as the parameter. This resulting equation is an integral of the geodesic equation and can be solved to give $t(\tau)$ and $x(\tau)$.]

Solution: So, for a time like geodesic we have

$$ds^2 = -d\tau^2 = -\frac{1}{t^2}dt^2 + \frac{1}{t^2}dx^2 \tag{9}$$

for real proper time $d\tau$. Using dots for differenciation with respect to τ this gives

$$-1 = \frac{1}{t^2}(\dot{x}^2 - \dot{t}^2) \tag{10}$$

Now, the corresponding Lagrangian $L = g_{ab}\dot{x}^a\dot{x}^b$ is

$$L = \frac{1}{t^2}(\dot{x}^2 - \dot{t}^2) \tag{11}$$

and, since this is independant of x one of the Euler-Lagrange equations is a conservation equation:

$$\frac{d}{d\tau}\frac{2\dot{x}}{t^2} = 0 \tag{12}$$

Integrating, this means that $\dot{x} = ct^2$ for some constant c. Substituting back into the proper distance formula (10)

$$\dot{t}^2 = t^2(1 + c^2t^2) \tag{13}$$

or

$$d\tau = \frac{dt}{t\sqrt{1+c^2t^2}} \tag{14}$$

and, using the integral given in the question, this means that

$$e^{2\tau} = \frac{\sqrt{1+c^2t^2}-1}{\sqrt{1+c^2t^2}+1} \tag{15}$$

where we have set the integration constant to zero, basically this is saying that $t = 0$ corresponds to $\tau = -\infty$. Solving for $\sqrt{1+c^2t^2}$ gives

$$\sqrt{1+c^2t^2} = -\frac{e^{\tau}+e^{-\tau}}{e^{\tau}-e^{-\tau}} \tag{16}$$

or

$$t = \frac{\pm 1}{c\sinh\tau} \tag{17}$$

4. (2) Poisson's fomulation of Newtonian gravity is

$$\begin{aligned} \nabla^2\phi &= 4\pi\rho \\ \mathbf{g} &= -\nabla\phi \end{aligned} \tag{18}$$

where ρ is the matter density, ϕ is the gravitational potential and $\mathbf{g}$ is the accelleration due to gravity. Show this gives the usual Newtonian formula for a point-like source.

Solution: The normal Newtonian formula is that the accelleration due to gravity in the field of a point like source of strength M is

$$\mathbf{g} = -\frac{M\mathbf{r}}{r^3} \tag{19}$$

Now, away from $r = 0$ we have

$$-\nabla\cdot\mathbf{g} = \nabla^2\phi = -M\nabla\frac{\mathbf{r}}{r^3} \tag{20}$$

and

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{1}{r^3} - \frac{3x^2}{r^5} \tag{21}$$

so

$$\frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial z}\frac{z}{r^3} = 0 \tag{22}$$

and $\rho = 0$ except at $r = 0$, in fact, for a point source we expect $\rho = \delta(x)\delta(y)\delta(z)M$ so, if S is a 2-sphere around the origin and B the ball it contains, we expect

$$\int\int\int_B \rho\, dxdydz = M \tag{23}$$

Now,

$$\int\int\int_B \nabla^2\phi dxdydz = -\int\int\int_B \nabla\cdot\mathbf{g}dxdydz = -\int\int_S \mathbf{g}\cdot\frac{\mathbf{r}}{r}r^2 d\Omega \tag{24}$$

by the Gauß theorem. Hence

$$\int\int\int_B \nabla^2\phi dxdydz = \int\int_S M d\Omega = 4\pi M \tag{25}$$

as required.

5. (1) What is the value of ϕ on the surface of the Earth.

   Solution: Well, $\mathbf{g} = -\nabla\mathbf{g}$ so

   $$\phi = \frac{M}{r} \tag{26}$$

   and in this case M is the mass of the earth and r is its radius. Rather than working in Planck units, notice the overall units, ϕ is a potential for accelleration, so it should be L^2T^{-2}, M is M and r is L so we need $L^3M^{-1}T^{-2}$, hence a G, that is,

   $$\phi = \frac{GM}{r} \tag{27}$$

   and the mass of the earth is 6.0×10^{24}kg, the radius of the earth is 6.4×10^6 m and $G = 6.7\times 10^{-11}$m^{3}kg^{-1}s^{-2} thus

   $$\phi = 7.1\times 10^{-28} \tag{28}$$

   I think.

6. (3) Find the Newtonian limit of Einstein gravity with cosmological constant.

   Solution: So, working with usual approximate solution from the notes, we have the Einstein equation

   $$\begin{aligned} R_{00} - \frac{1}{2}Rg_{00} + \Lambda g_{00} &= 8\pi\rho \\ R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} &\approx 0 \end{aligned} \tag{29}$$

   Now, we can use the spatial equation to eliminate R:

   $$R = g^{00}R_{00} + g^{ij}R_{ij} = -R_{00} + \frac{3}{2}R - 3\Lambda \tag{30}$$

   where we have used $g^{ij}g_{ij} = 3$. Now $R_{00} = \nabla^2\phi$ and $R = 2R_{00} - 6\Lambda$. Finally $g_{00} = -1 - 2\phi$ so we get

   $$(\nabla^2 - 4\Lambda)\phi = 4\pi\rho + \frac{1}{2}\Lambda \tag{31}$$