1. (2) Find an expression for
\[ \nabla_a \nabla_a T^a_b - \nabla_a \nabla_b T^a_a \] (1)
in terms of the Riemann tensor.

**Solution:** So, the thing to remember here is that \( \nabla_a T^a_b \) is a three-indexed tensor and must be differentiated accordingly. Hence

\[ \nabla_a (\nabla_a T^a_b) = \partial_b (\nabla_a T^a_b) + \Gamma^a_{ab} (\nabla_a T^a_c) - \Gamma^a_{ac} (\nabla_a T^a_b) - \Gamma^a_{bc} (\nabla_a T^a_a) \] (2)

We also know that

\[ \nabla_a T^a_b = \partial_b T^a_a + \Gamma^a_{ad} T^d_b - \Gamma^a_{db} T^d_a \] (3)

Expanding out the whole lot

\[ \nabla_a (\nabla_a T^a_b) = \left( \Gamma^a_{cd} + \Gamma^a_{cf} \Gamma^f_d \right) T^d_b + \left( -\Gamma^a_{bc} + \Gamma^a_{cd} \Gamma^c_d \right) T^d_b + \text{terms symmetric in } c \text{ and } d \] (4)

where we haven't written out the terms symmetric in \( c \) and \( d \) because we know they will cancel when we subtract \( \nabla_d (\nabla_a T^a_b) \). Now,

\[ \nabla_a (\nabla_a T^a_b) - \nabla_a (\nabla_a T^a_b) = \left( \Gamma^a_{cd} - \Gamma^a_{cd} - \Gamma^a_{bc} + \Gamma^a_{cd} \Gamma^b_d \right) T^d_b + \left( -\Gamma^a_{bc} + \Gamma^a_{cd} \Gamma^c_d - \Gamma^a_{bc} \Gamma^d_c \right) T^d_b \]
\[ \quad = R^a_{cd} T^d_b + R_{cdad} T^d_b \] (5)

2. (2) Calculate the usual metric on the surface of a sphere by considering a radius \( r \) sphere \( x^2 + y^2 + z^2 = r^2 \) embedded in three-dimensional flat space \( ds^2 = dx^2 + dy^2 + dz^2 \). To do this, change to spherical polar coordinates:

\[ x = r \cos \phi \sin \theta \]
\[ y = r \sin \phi \sin \theta \]
\[ z = r \cos \theta \] (6)

and then set \( dr = 0 \) to restrict to the surface of the sphere.

Well, this is just a change of coordinates and we know how to do this:

\[ g_{ab} \rightarrow g_{uv} = A_a^u A_v^b g_{ab} \] (7)

where

\[ A_a^b = \frac{\partial x^a}{\partial x^v} \] (8)

So let's use \( (x, y, z) = [x^u] \) and \( (r, \theta, \phi) = [x^v] \) and work out the derivatives. First \( x \):

\[ \frac{\partial x}{\partial r} = \cos \phi \sin \theta \]
\[ \frac{\partial x}{\partial \theta} = r \cos \phi \cos \theta \]
\[ \frac{\partial x}{\partial \phi} = -r \sin \phi \sin \theta \] (9)

then \( y \):

\[ \frac{\partial y}{\partial r} = \sin \phi \sin \theta \]
\[ \frac{\partial y}{\partial \theta} = r \sin \phi \cos \theta \]
\[ \frac{\partial y}{\partial \phi} = r \cos \phi \sin \theta \] (10)

and finally \( z \):

\[ \frac{\partial z}{\partial r} = \cos \theta \]
\[ \frac{\partial z}{\partial \theta} = -r \sin \theta \]
\[ \frac{\partial z}{\partial \phi} = 0 \] (11)

So, using the notation \( g_{uv} \) for \( g_{11}, g_{rr} \) for \( g_{12} \), and so on we have

\[ g_{rr} = (\cos \phi \sin \theta)^2 + (\sin \phi \sin \theta)^2 + (\cos \theta)^2 = 1 \] (12)

and

\[ g_{uu} = (r \cos \phi \cos \theta)^2 + (r \sin \phi \cos \theta)^2 + (-r \sin \theta)^2 = r^2 \] (13)

and

\[ g_{\phi \phi} = (-r \sin \phi \sin \theta)^2 + (r \cos \phi \sin \theta)^2 = r^2 \sin^2 \theta \] (14)

You should also check that the cross terms are all zero, for example

\[ g_{r\theta} = A_r^u A_\theta^v + A_r^v A_\theta^u \]
Putting all this together we get

\[ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

and, so, on the sphere \( dr = 0 \) and

\[ ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

(17)

3. (4) Find the curvature on a two-dimensional hyperboloid:

\[ r^2 - x^2 - y^2 = r^2 \]

embedded in Minkowski space:

\[ ds^2 = -dt^2 + dx^2 + dy^2 \]

(19)

In other words, change to hyperbolic coordinates

\[
\begin{align*}
x &= r \cos \phi \sinh \eta \\
y &= r \sin \phi \sinh \eta \\
t &= r \cosh \eta
\end{align*}
\]

(20)

and then restrict to the surface of the hyperboloid by setting \( dr = 0 \). This gives the metric on the surface of the embedded hyperboloid, now calculate its Ricci scalar.

Solution: To do the change of coordinates, we could work out all the \( A \) factors as before, a notationally simpler route is used here, we calculate \( dx, dy \) and \( dz \) by differentiating, for example,

\[ dx = \sinh \eta \sin \phi dr + r \cosh \eta \sin \phi d\eta + r \sin \eta \cos \phi d\phi. \]

(21)

Next, we calculate \( ds^2 = -dt^2 + dx^2 + dy^2 \), this is just a matter of multiplying out and gathering together. We get

\[ ds^2 = -dt^2 + r^2 d\eta^2 + r^2 \sin^2 \eta d\phi^2 \]

(22)

and for fixed \( r \) we have \( dr = 0 \) and so the metric on the hyperboloid is

\[ ds^2 = r^2 d\eta^2 + r^2 \sin^2 \eta d\phi^2 \]

(23)

giving metric

\[
[g_{ab}] = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \eta \end{pmatrix}
\]

(24)

and

\[
[g^{ab}] = \frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1/ \sinh^2 \eta \end{pmatrix}
\]

(25)

Next, we work out the connection coefficients:

\[ \Gamma^a_{bc} = \frac{1}{2} g^{ad} \partial_d (g_{bc} + g_{bd} - g_{cd}) \]

(26)

Hence, since \( [g^{ab}] \) is diagonal, we have

\[ \Gamma^a_{bc} = \frac{1}{2 r^2} (\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc}) \]

(27)

Now, the only non-constant term in the metric is \( g_{\phi\phi} \) so the only non-zero connection coefficient with superscript \( \eta \) is

\[ \Gamma^\eta_{\phi\phi} = \frac{1}{2 r^2 \sinh^2 \eta} (\partial_c g_{\phi\phi} + \partial_b g_{\phi\phi} - \partial_a g_{\phi\phi}) \]

(29)

and the only non-zero possibilities are

\[ \Gamma^\phi_{\eta\phi} = \Gamma^\phi_{\eta\phi} = \frac{1}{2 r^2 \sinh^2 \eta} (\partial_c g_{\phi\phi}) = \coth \eta \]

(30)

Next, we work out the Riemann tensor

\[ R_{abc}^d = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^e_{ac} \Gamma^d_{be} - \Gamma^e_{bc} \Gamma^d_{ae} \]

(31)

and, because \( R_{abcd} = -R_{bacd} = -R_{abdc} \) and the metric is diagonal, there is only independent nonzero component of the Riemann tensor and it is sufficient to calculate \( R_{\phi\eta\eta} \).

Now,

\[ R_{\phi\eta\eta} = \partial_\phi \Gamma^\eta_{\eta\phi} - \partial_\eta \Gamma^\eta_{\phi\phi} + \Gamma^\lambda_{\eta\phi} \Gamma^\eta_{\lambda\phi} - \Gamma^\lambda_{\phi\phi} \Gamma^\eta_{\lambda\phi} \]

(32)

Only the first and last term are non-zero, giving

\[ R_{\phi\eta\eta} = \partial_\phi \partial_\eta g_{\phi\phi} - \partial_\phi g_{\phi\phi} \partial_\eta \]

\[ = -\sinh^2 \eta - \cosh^2 \eta - \cosh^2 \eta = -\sinh^2 \eta \]

(33)
Finally,

\[ R_{\phi\phi} = R_{\text{outside}} = R_{\text{inside}} = -\sinh^2 \eta \]  

and

\[ R_{\text{inside}} = g_{\text{pp}} R_{\phi\phi} = -r^2 \sinh^2 \eta \]  

so

\[ R_{\text{pp}} = g_{\text{pp}} R_{\text{inside}} = -1 \]  

Putting all this together we get

\[ R = g^{pp} R_{pp} + g^{\phi\phi} R_{\phi\phi} = -\frac{2}{r^2} \]