442 Tutorial Sheet 1b[1]

1. (2) Find an expression for
$$\nabla_c \nabla_d T^a_{\ b} - \nabla_d \nabla_c T^a_{\ b} \tag{1}$$
in terms of the Riemann tensor.

   Solution: So, the thing to remember here is that $\nabla_d T^a_{\ b}$ is a three-indexed tensor and must be differenciated accordingly. Hence
$$\nabla_c \left(\nabla_d T^a_{\ b}\right) = \partial_c \left(\nabla_d T^a_{\ b}\right) + \Gamma^a_{ce} \left(\nabla_d T^e_{\ b}\right) - \Gamma^e_{cd} \left(\nabla_e T^a_{\ b}\right) - \Gamma^e_{cb} \left(\nabla_d T^a_{\ e}\right) \tag{2}$$
   We also know that
$$\nabla_d T^a_{\ b} = \partial_d T^a_{\ b} + \Gamma^a_{ed} T^e_{\ b} - \Gamma^e_{db} T^a_{\ e} \tag{3}$$
   Expanding out the whole lot
$$\begin{aligned} \nabla_c \left(\nabla_d T^a_{\ b}\right) &= \left(\Gamma^a_{ed,c} + \Gamma^a_{cf}\Gamma^f_{ed}\right) T^e_{\ b} + \left(-\Gamma^e_{db,c} + \Gamma^f_{cb}\Gamma^e_{df}\right) T^a_{\ e} \\ &\quad + \text{terms symmetric in } c \text{ and } d \end{aligned} \tag{4}$$
   where we haven't written out the terms symmetric in c and d because we know they will cancel when we subtract $\nabla_d \left(\nabla_e T^a_{\ b}\right)$. Now,
$$\begin{aligned} \nabla_c \left(\nabla_d T^a_{\ b}\right) - \nabla_d \left(\nabla_e T^a_{\ b}\right) &= \left(\Gamma^a_{ed,c} - \Gamma^a_{ec,d} + \Gamma^a_{cf}\Gamma^f_{ed} - \Gamma^a_{df}\Gamma^f_{ec}\right) T^e_{\ b} \\ &\quad + \left(\Gamma^e_{cb,d} - \Gamma^e_{db,c} + \Gamma^f_{cb}\Gamma^e_{df} - \Gamma^f_{db}\Gamma^e_{cf}\right) T^a_{\ e} \\ &= R_{ecd}{}^a T^e_{\ b} + R_{cdb}{}^e T^a_{\ e} \end{aligned} \tag{5}$$

2. (2) Calculate the usual metric on the surface of a sphere by considering a radius r sphere $x^2 + y^2 + z^2 = r^2$ embedded in three-dimensional flat space $ds^2 = dx^2 + dy^2 + dz^2$. To do this, change to spherical polar coördinates:
$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta \end{aligned} \tag{6}$$
   and then set $dr = 0$ to restrict to the surface of the sphere.

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Well, this is just a change of coördinates and we know how to this:

$$g_{ab} \mapsto g_{a'b'} = A^a_{a'} A^b_{b'} g_{ab} \tag{7}$$

where

$$A^a_{a'} = \frac{\partial x^a}{\partial x^{a'}} \tag{8}$$

So lets use $(x, y, z) = [x^a]$ and $(r, \theta, \phi) = [x^{a'}]$ and work out the derivatives. First x:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\phi \sin\theta \\ \frac{\partial x}{\partial \theta} &= r\cos\phi\cos\theta \\ \frac{\partial x}{\partial \phi} &= -r\sin\phi\sin\theta \end{aligned} \tag{9}$$

then y

$$\begin{aligned} \frac{\partial y}{\partial r} &= \sin\phi \sin\theta \\ \frac{\partial y}{\partial \theta} &= r\sin\phi\cos\theta \\ \frac{\partial y}{\partial \phi} &= r\cos\phi\sin\theta \end{aligned} \tag{10}$$

and finally z

$$\begin{aligned} \frac{\partial z}{\partial r} &= \cos\theta \\ \frac{\partial y}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial \phi} &= 0 \end{aligned} \tag{11}$$

So, using the notation g_{xx} for g_{11}, g_{rr} for $g_{1'1'}$ and so on we have

$$g_{rr} = (\cos\phi\sin\theta)^2 + (\sin\phi\sin\theta)^2 + (\cos\theta)^2 = 1 \tag{12}$$

and

$$g_{\theta\theta} = (r\cos\phi\cos\theta)^2 + (r\sin\phi\cos\theta)^2 + (-r\sin\theta)^2 = r^2 \tag{13}$$

and

$$g_{\phi\phi} = (-r\sin\phi\sin\theta)^2 + (r\cos\phi\sin\theta)^2 = r^2\sin^2\theta \tag{14}$$

You should also check that the cross terms are all zero, for example

$$g_{r\theta} = A^x_r A^x_\theta + A^y_r A^y_\theta + A^z_r A^z_\theta$$

$$= \quad r\cos^2\phi\sin\theta\cos\theta + r\sin^2\phi\cos\theta\sin\theta - r\sin\theta\cos\theta \tag{15}$$

Putting all this together we get

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{16}$$

and, so, on the sphere $dr = 0$ and

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{17}$$

3. (4) Find the curvature on a two-dimensional hyperboloid:

$$t^2 - x^2 - y^2 = r^2 \tag{18}$$

embedded in Minkowski space:

$$ds^2 = -dt^2 + dx^2 + dy^2 \tag{19}$$

In other words, change to hyperbolic coördinates

$$\begin{aligned} x &= r\cos\phi\sinh\eta \\ y &= r\sin\phi\sinh\eta \\ t &= r\cosh\eta \end{aligned} \tag{20}$$

and then restrict to the surface of the hyperboloid by setting $dr = 0$. This gives the metric on the surface of the embedded hyperboloid, now calculate its Ricci scalar.

Solution: To do the change of coördinates, we could work out all the A factors as before, a notationally simpler route is used here, we calculate dx, dy and dz by differenciating, for example,

$$dx = \sinh\eta\sin\phi dr + r\cosh\eta\sin\phi d\eta + r\sinh\eta\cos\phi d\phi. \tag{21}$$

Next, we calculate $ds^2 = -dt^2 + dx^2 + dy^2$, this is just a matter of multiplying out and gathering together. We get

$$ds^2 = -dr^2 + r^2 d\eta^2 + r^2\sinh^2\eta d\phi^2 \tag{22}$$

and for fixed r we have $dr = 0$ and so the metric on the hyperboloid is

$$ds^2 = r^2 d\eta^2 + r^2\sinh^2\eta d\phi^2 \tag{23}$$

giving metric

$$[g_{ab}] = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\eta \end{pmatrix} \tag{24}$$

and

$$[g^{ab}] = \frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1/\sinh^2\eta \end{pmatrix} \tag{25}$$

Next, we work out the connection coefficients:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}\right) \tag{26}$$

Hence, since $[g^{ab}]$ is diagonal, we have

$$\Gamma^\eta_{ab} = \frac{1}{2r^2} \left(\partial_a g_{\eta b} + \partial_b g_{a\eta} - \partial_\eta g_{ab}\right) \tag{27}$$

Now, the only non-constant term in the metric is $g_{\phi\phi}$ so the only non-zero connection coefficient with superscript η is

$$\Gamma^\eta_{\phi\phi} = \frac{1}{2r^2} \left(-\partial_\eta g_{\phi\phi}\right) = -\frac{1}{2r^2} \partial_\eta \left(r^2 \sinh^2\eta\right) = -\cosh\eta \sinh\eta \tag{28}$$

Similarily

$$\Gamma^\phi_{ab} = \frac{1}{2r^2 \sinh^2\eta} \left(\partial_a g_{\phi b} + \partial_b g_{a\phi} - \partial_\phi g_{ab}\right) \tag{29}$$

and the only non-zero possibilities are

$$\Gamma^\phi_{\eta\phi} = \Gamma^\phi_{\phi\eta} = \frac{1}{2r^2 \sinh^2\eta} \left(\partial_\eta g_{\phi\phi}\right) = \coth\eta \tag{30}$$

Next, we work out the Riemann tensor

$$R_{abc}{}^d = \partial_b \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^f_{ac}\Gamma^d_{bf} - \Gamma^f_{bc}\Gamma^d_{af} \tag{31}$$

and, because $R_{abcd} = -R_{bacd} = -R_{abdc}$ and the metric is diagonal, there is only independant nonzero component of the Riemann tensor and it is sufficient to calculate $R_{\phi\eta\phi\eta}$. Now,

$$R_{\phi\eta\phi}{}^\eta = \partial_\eta \Gamma^\eta_{\phi\phi} - \partial_\phi \Gamma^\eta_{\phi\eta} + \Gamma^f_{\phi\phi}\Gamma^\eta_{\eta f} - \Gamma^f_{\phi\eta}\Gamma^\eta_{\phi f} \tag{32}$$

Only the first and last term are non-zero, giving

$$\begin{aligned} R_{\phi\eta\phi}{}^\eta &= -\partial_\eta \cosh\eta \sinh\eta + coth\eta \sinh\eta \cosh\eta \\ &= -\sinh^2\eta - \cosh^2\eta + cosh^2\eta = -\sinh^2\eta \end{aligned} \tag{33}$$

Finally,

$$R_{\phi\phi} = R_{\phi a\phi}{}^{a} = R_{\phi\eta\phi}{}^{\eta} = -\sinh^2\eta \tag{34}$$

and

$$R_{\phi\eta\phi\eta} = g_{\eta a} R_{\phi\eta\phi}{}^{a} = -r^2\sinh^2\eta \tag{35}$$

so

$$R_{\eta\eta} = g^{\phi\phi} R_{\phi\eta\phi\eta} = -1 \tag{36}$$

Putting all this together we get

$$R = g^{\eta\eta} R_{\eta\eta} + g^{\phi\phi} R_{\phi\phi} = -\frac{2}{r^2} \tag{37}$$