1. (2) Find an expression for $\nabla_c \nabla_d T^a_b - \nabla_d \nabla_c T^a_b$ in terms of the Riemann tensor.

*Solution:* So, the thing to remember here is that $\nabla_d T^a_b$ is a three-indexed tensor and must be differentiated accordingly. Hence

$$\nabla_c (\nabla_d T^a_b) = \partial_c (\nabla_d T^a_b) + \Gamma^a_{ce} (\nabla_d T^e_b) - \Gamma^e_{cd} (\nabla_e T^a_b) - \Gamma^e_{cb} (\nabla_d T^a_e)$$

We also know that

$$\nabla_d T^a_b = \partial_d T^a_b + \Gamma^a_{ed} T^e_b - \Gamma^e_{db} T^a_e$$

Expanding out the whole lot

$$\nabla_c (\nabla_d T^a_b) = \left( \Gamma^a_{ed,c} + \Gamma^a_{cf} \Gamma^f_{ed} \right) T^e_b + \left( -\Gamma^e_{db,c} + \Gamma^f_{eb} \Gamma^e_{df} \right) T^a_e$$

where we haven’t written out the terms symmetric in $c$ and $d$ because we know they will cancel when we subtract $\nabla_d (\nabla_c T^a_b)$. Now,

$$\nabla_c (\nabla_d T^a_b) - \nabla_d (\nabla_c T^a_b) = \left( \Gamma^a_{ed,c} - \Gamma^a_{ec,d} + \Gamma^a_{cf} \Gamma^f_{ed} - \Gamma^f_{df} \Gamma^e_{ec} \right) T^e_b$$

$$+ \left( \Gamma^e_{db,c} - \Gamma^e_{dc,b} + \Gamma^f_{eb} \Gamma^e_{df} - \Gamma^f_{df} \Gamma^e_{eb} \right) T^a_e$$

$$= R^a_{eb,c} T^e_b + R^a_{db,c} T^e_b$$

2. (2) Calculate the usual metric on the surface of a sphere by considering a radius $r$ sphere $x^2 + y^2 + z^2 = r^2$ embedded in three-dimensional flat space $ds^2 = dx^2 + dy^2 + dz^2$. To do this, change to spherical polar coordinates:

$$x = r \cos \phi \sin \theta$$
$$y = r \sin \phi \sin \theta$$
$$z = r \cos \theta$$

and then set $dr = 0$ to restrict to the surface of the sphere.
Well, this is just a change of coordinates and we know how to this:

\[ g_{ab} \mapsto g_{a'b'} = A^a_{\alpha} A^b_{\beta} g_{ab} \]  

(7)

where

\[ A^a_{\alpha} = \frac{\partial x^a}{\partial x^\alpha} \]  

(8)

So lets use \((x, y, z) = [x^a]\) and \((r, \theta, \phi) = [x^{a'}]\) and work out the derivatives. First \(x\):

\[
\begin{align*}
\frac{\partial x}{\partial r} &= \cos \phi \sin \theta \\
\frac{\partial x}{\partial \theta} &= r \cos \phi \cos \theta \\
\frac{\partial x}{\partial \phi} &= -r \sin \phi \sin \theta
\end{align*}
\]

then \(y\):

\[
\begin{align*}
\frac{\partial y}{\partial r} &= \sin \phi \sin \theta \\
\frac{\partial y}{\partial \theta} &= r \sin \phi \cos \theta \\
\frac{\partial y}{\partial \phi} &= r \cos \phi \sin \theta
\end{align*}
\]

and finally \(z\):

\[
\begin{align*}
\frac{\partial z}{\partial r} &= \cos \theta \\
\frac{\partial z}{\partial \theta} &= -r \sin \theta \\
\frac{\partial z}{\partial \phi} &= 0
\end{align*}
\]

(11)

So, using the notation \(g_{xx}\) for \(g_{11}\), \(g_{rr}\) for \(g_{11'}\) and so on we have

\[ g_{rr} = (\cos \phi \sin \theta)^2 + (\sin \phi \sin \theta)^2 + (\cos \theta)^2 = 1 \]  

(12)

and

\[ g_{\theta\theta} = (r \cos \phi \cos \theta)^2 + (r \sin \phi \cos \theta)^2 + (-r \sin \theta)^2 = r^2 \]  

(13)

and

\[ g_{\phi\phi} = (-r \sin \phi \sin \theta)^2 + (r \cos \phi \sin \theta)^2 = r^2 \sin^2 \theta \]  

(14)

You should also check that the cross terms are all zero, for example

\[ g_{r\theta} = A^x_r A^x_\theta + A^x_\theta A^y_r + A^x_r A^y_\theta \]
Putting all this together we get

\[ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

(16)

and, so, on the sphere \( dr = 0 \) and

\[ ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

(17)

3. (4) Find the curvature on a two-dimensional hyperboloid:

\[ t^2 - x^2 - y^2 = r^2 \]  

(18)

embedded in Minkowski space:

\[ ds^2 = -dt^2 + dx^2 + dy^2 \]  

(19)

In other words, change to hyperbolic coordinates

\[
\begin{align*}
x &= r \cos \phi \sinh \eta \\
y &= r \sin \phi \sinh \eta \\
t &= r \cosh \eta
\end{align*}
\]

(20)

and then restrict to the surface of the hyperboloid by setting \( dr = 0 \). This gives the metric on the surface of the embedded hyperboloid, now calculate its Ricci scalar.

\textbf{Solution:} To do the change of coördinates, we could work out all the \( A \) factors as before, a notationally simpler route is used here, we calculate \( dx, dy \) and \( dz \) by differenciating, for example,

\[ dx = \sinh \eta \sin \phi dr + r \cosh \eta \sin \phi d\eta + r \sinh \eta \cos \phi d\phi. \]  

(21)

Next, we calculate \( ds^2 = -dt^2 + dx^2 + dy^2 \), this is just a matter of multiplying out and gathering together. We get

\[ ds^2 = -dr^2 + r^2 d\eta^2 + r^2 \sinh^2 \eta d\phi^2 \]  

(22)

and for fixed \( r \) we have \( dr = 0 \) and so the metric on the hyperboloid is

\[ ds^2 = r^2 d\eta^2 + r^2 \sinh^2 \eta d\phi^2 \]  

(23)
giving metric
\[ [g_{ab}] = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \eta \end{pmatrix} \] (24)

and
\[ [g^{ab}] = \frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & 1/ \sinh^2 \eta \end{pmatrix} \] (25)

Next, we work out the connection coefficients:
\[ \Gamma^c_{ab} = \frac{1}{2} g^{cd} (\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}) \] (26)

Hence, since \([g^{ab}]\) is diagonal, we have
\[ \Gamma^\eta_{ab} = \frac{1}{2r^2} (\partial_a g_{b\phi} + \partial_b g_{a\phi} - \partial_{\phi} g_{ab}) \] (27)

Now, the only non-constant term in the metric is \(g_{\phi\phi}\) so the only non-zero connection coefficient with superscript \(\eta\) is
\[ \Gamma^\eta_{\phi\phi} = \frac{1}{2r^2} (-\partial_{\eta} g_{\phi\phi}) = -\frac{1}{2r^2} \partial_{\eta} \left( r^2 \sinh^2 \eta \right) = -\cosh \eta \sinh \eta \] (28)

Similarly
\[ \Gamma^\phi_{ab} = \frac{1}{2r^2 \sinh^2 \eta} (\partial_a g_{b\phi} + \partial_b g_{a\phi} - \partial_{\phi} g_{ab}) \] (29)

and the only non-zero possibilities are
\[ \Gamma^\phi_{\eta\phi} = \Gamma^\phi_{\phi\eta} = \frac{1}{2r^2 \sinh^2 \eta} (\partial_{\eta} g_{\phi\phi}) = \coth \eta \] (30)

Next, we work out the Riemann tensor
\[ R^d_{abc} = \partial_d \Gamma^d_{ac} - \partial_a \Gamma^d_{bc} + \Gamma^f_{ac} \Gamma^d_{bf} - \Gamma^f_{bc} \Gamma^d_{af} \] (31)

and, because \(R^{abcd} = -R^{bacd} = -R^{abdc}\) and the metric is diagonal, there is only independent nonzero component of the Riemann tensor and it is sufficient to calculate \(R_{\phi\eta\phi\eta}\). Now,
\[ R_{\phi\eta\phi\eta} = \partial_\eta \Gamma^\eta_{\phi\phi} - \partial_\phi \Gamma^\eta_{\phi\eta} + \Gamma^f_{\phi\eta} \Gamma^\eta_{\phi f} - \Gamma^f_{\phi\phi} \Gamma^\eta_{\phi f} \] (32)

Only the first and last term are non-zero, giving
\[ R_{\phi\eta\phi} = -\partial_\eta \cosh \eta \sinh \eta + \coth \eta \sinh \eta \cosh \eta = -\sinh^2 \eta - \cosh^2 \eta + \cosh^2 \eta = -\sinh^2 \eta \] (33)
Finally,

\[ R_{\phi \phi} = R_{\phi a \phi}^a = R_{\phi \eta \phi}^\eta = - \sinh^2 \eta \]  

(34)

and

\[ R_{\phi \eta \phi \eta} = g_{\eta a} R_{\phi \eta \phi}^a = - r^2 \sinh^2 \eta \]  

(35)

so

\[ R_{\eta \eta} = g^{\phi \phi} R_{\phi \eta \phi \eta} = -1 \]  

(36)

Putting all this together we get

\[ R = g^{\eta \eta} R_{\eta \eta} + g^{\phi \phi} R_{\phi \phi} = - \frac{2}{r^2} \]  

(37)