442 Tutorial Sheet 1 Solutions[1]

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1. (1) Show that $T^{ab}S_b$ is a tensor, where T and S are tensors.

   Solution: So, since they are tensors, if
$$x^a \to x^{a'} \tag{1}$$
   then
$$\begin{aligned} T^{ab} &\to T^{a'b'} = A^{a'}_a A^{b'}_b T^{ab} \\ S_a &\to S_{a'} = A^a_{a'} S_a \end{aligned} \tag{2}$$
   hence
$$T^{ab}S_b \to T^{a'b'}S'_b = A^{a'}_a A^{b'}_b T^{ab} A^c_{b'} S_c \tag{3}$$
   but we know that
$$A^{b'}_b A^c_{b'} = \delta^c_b \tag{4}$$
   so
$$T^{ab}S_b \to T^{a'b'}S'_b = A^{a'}_a T^{ab} S_b \tag{5}$$
   as required.

2. (1) Show that $T^{(ab)}V_{[a|c|b]}$ vanishes, where T and V are tensors.

   Solution: Well, remember that
$$V_{[a|b|c]} = \frac{1}{2}(V_{abc} - V_{cba}) \tag{6}$$
   and so $V_{[a|b|c]} = -V_{[c|b|a]}$. Furthermore
$$T^{(ab)} = \frac{1}{2}(T^{ab} + T^{ba}) \tag{7}$$
   and $T^{(ab)} = T^{(ba)}$. Thus
$$T^{(ab)}V_{[a|c|b]} = -T^{(ab)}V_{[b|c|a]} \tag{8}$$

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and changing dummy indexes and then using the symmetry of $T^{(ab)}$
$$T^{(ab)}V_{[a|c|b]} = -T^{(ba)}V_{[a|c|b]} = -T^{(ab)}V_{[a|c|b]} \tag{9}$$

3. (1) Show that $T^{ab}{}_e = T^{abc}{}_{de}\delta^d_c$ is a tensor.

   Solution: So, this is very long to write out, but here goes, under
$$x^a \to x^{a'} \tag{10}$$
   we have
$$T^{abc}{}_{de} \to T^{a'b'c'}{}_{d'e'} = A^{a'}_a A^{b'}_b A^{c'}_c A^d_{d'} A^e_{e'} T^{abc}{}_{de} \tag{11}$$
   Thus
$$T^{abd}{}_{de} \to T^{a'b'd'}{}_{d'e'} = A^{a'}_a A^{b'}_b A^{d'}_c A^d_{d'} A^e_{e'} T^{abc}{}_{de} = A^{a'}_a A^{b'}_b A^e_{e'} T^{abd}{}_{de} \tag{12}$$
   where we have used
$$A^{d'}_c A^d_{d'} = \delta^d_c \tag{13}$$

   Alternatively, observer that δ^b_a is a tensor, say $T^b_a = \delta^b_a$ in some coördinate system, ther under a coördinate transformation
$$T^b_a \mapsto T^{b'}_{a'} = A^a_{a'} A^{b'}_b T^b_a = A^a_{a'} A^{b'}_b \delta^b_a = A^a_{a'} A^{b'}_a = \delta^{b'}_{a'} \tag{14}$$
   Thus, the Kronecker delta is a tensor. The contraction and multiplication properties c tensors can now be invoked.

4. (1) Show that $T^{ab} = -T^{ba}$ in one coördinate system implies that $T^{a'b'} = -T^{b'a'}$ i another coördinate system.

   Solution: Well
$$T^{a'b'} = A^{a'}_a A^{b'}_b T^{ab} \tag{15}$$
   and hence
$$T^{b'a'} = A^{b'}_a A^{a'}_b T^{ab} \tag{16}$$
   then changin the dummy indices and then using the antisymmetry of T we have
$$T^{b'a'} = A^{b'}_b A^{a'}_a T^{ba} = -A^{b'}_b A^{a'}_a T^{ab} = -T^{a'b'} \tag{17}$$

Solution: Well

$$T^{a'b'} = A^{a'}_a A^{b'}_b T^{ab} \tag{18}$$

and hence

$$T^{b'a'} = A^{b'}_a A^{a'}_b T^{ab} \tag{19}$$

then changin the dummy indices and then using the antisymmetry of T we have

$$T^{b'a'} = A^{b'}_b A^{a'}_a T^{ba} = -A^{b'}_b A^{a'}_a T^{ab} = -T^{a'b'} \tag{20}$$

5. (3) Write $\triangle f$ in two-dimensional polar coordinates using the torsion free metric connection..

Solution: So,

$$\triangle f = D_a D^a f = D_a \partial^a f \tag{21}$$

where we have used the fact that f is a scalar. Now, from the formula for the covariant derivative

$$\triangle f = \partial_a \partial^a f + \Gamma^a_{ab}\partial^b f \tag{22}$$

Hence, we need to work out the connection coefficients with the up index equal to the first down index. Now, for polar coördinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{23}$$

or

$$[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{24}$$

and

$$[g^{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{25}$$

Using the fact that this is diagonal, we have

$$\begin{aligned} \Gamma^r_{rr} &= \frac{1}{2} g_{rr,r} = 0 \\ \Gamma^r_{r\theta} &= \frac{1}{2} g_{rr,\theta} = 0 \\ \Gamma^\theta_{\theta\theta} &= \frac{1}{2r^2} g_{\theta\theta,\theta} = 0 \\ \Gamma^\theta_{r\theta} &= \frac{1}{2r^2} g_{\theta\theta,r} = \frac{1}{r} \end{aligned} \tag{26}$$

So the only nonzero entry is $\Gamma^\theta_{r\theta}$ and so we conclude

$$\triangle f = \frac{\partial^2}{\partial r^2} f + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} f + \frac{1}{r}\frac{\partial}{\partial r} f \tag{27}$$

6. (3) Show that torsion is a tensor.

Solution: So,

$$\text{det}\, g_{a'b'} = \text{det}\left(A^a_{a'} A^b_{b'} g_{a'b'}\right) = (\text{det}\, A^a_{a'})^2 \text{det}\, g_{ab} \tag{28}$$

Solution: So, to recall, the transformation property of the connection coefficients is

$$\Gamma^a_{bc} \to \Gamma^{a'}_{b'c'} = A^b_{b'} A^c_{c'} A^{a'}_a \Gamma^a_{bc} - A^b_{b'} A^c_{c'} (\partial_b A^{a'}_c) \tag{29}$$

Now, torsion is the anti-symmetric part of the connection

$$T^a_{bc} = \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right) \tag{30}$$

Using the tranformation law above, this means that

$$\begin{aligned} 2T^{a'}_{b'c'} &= A^b_{b'} A^c_{c'} A^{a'}_a \Gamma^a_{bc} - A^b_{b'} A^c_{c'} (\partial_b A^{a'}_c) - A^b_{c'} A^c_{b'} A^{a'}_a \Gamma^a_{bc} + A^b_{c'} A^c_{b'} (\partial_b A^{a'}_c) \\ &= A^b_{b'} A^c_{c'} A^{a'}_a 2T^a_{bc} + A^b_{c'} A^c_{b'} (\partial_b A^{a'}_c) - A^b_{b'} A^c_{c'} (\partial_b A^{a'}_c) \end{aligned} \tag{31}$$

Now, expanding out the notation,

$$\partial_b A^{a'}_c = \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} \tag{32}$$

is symmetric in b and c, this allows us to cancel the two non-tensor terms in the transformation law, proving the result.

7. (1) Find the transformation law for detg_{ab}.

Solution: So

$$\text{det}\, g_{a'b'} = \text{det}\left(A^a_{a'} A^b_{b'} g_{a'b'}\right) = (\text{det}\, A^a_{a'})^2 \text{det}\, g_{ab} \tag{33}$$

8. (3) Show that $\nabla_a g^{bc} = 0$ for a torsion free metric connection.

Solution: First, from the Leibnitz rule

$$\nabla_a \left(g^{bc} g_{cd}\right) = \nabla_a \delta^b_d$$

$$= \left(\nabla_a g^{bc}\right) g_{cd} + g^{bc} \nabla_a g_{cd} \tag{34}$$

and $\nabla_a g_{cd} = 0$ for a metric connection. Furthermore, since g^{bc} is defined as the inverse of g_{cd} we must be assuming that the metric is invertible and so, we need only to show $\nabla_a \delta^b_d = 0$. From the action of the covariant derivative on a (1,1) tensor we know

$$\nabla_a \delta^b_d = \partial_a \delta^b_d + \Gamma^d_{ae} \delta^e_b - \Gamma^e_{ab} \delta^d_e \tag{35}$$

but, the first term is zero since the delta is constant and the other two terms cancel.