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1. (1) Show that $T^{ab}S_b$ is a tensor, where $T$ and $S$ are tensors.

Solution: So, since they are tensors, if

$$x^a \rightarrow x'^a$$

then

$$T^{ab} \rightarrow T'^{a'b'} = A^a_{a'} A^b_{b'} T^{ab} \quad (1)$$

$$S_a \rightarrow S_{a'} = A^a_{a'} S_a \quad (2)$$

hence

$$T^{ab}S_b \rightarrow T'^{a'b'}S_{b'} = A^a_{a'} A^b_{b'} T^{ab} A^b_{b'} S_{b'} \quad (3)$$

but we know that

$$A^a_{a'} A^b_{b'} = \delta^a_{b'} \quad (4)$$

so

$$T^{ab}S_b \rightarrow T'^{a'b'}S_{b'} = A^a_{a'} T^{ab} S_{b'} \quad (5)$$

as required.

2. (1) Show that $T^{(ab)}V_{[a|b]}$ vanishes, where $T$ and $V$ are tensors.

Solution: Well, remember that

$$V_{[a|b]} = \frac{1}{2} (V_{abc} - V_{cba}) \quad (6)$$

and so $V_{[a|b]} = -V_{[b|a]}$. Furthermore

$$T^{(ab)} = \frac{1}{2} (T^{ab} + T^{ba}) \quad (7)$$

and $T^{(ab)} = T^{(ba)}$. Thus

$$T^{(ab)}V_{[a|b]} = -T^{(ab)}V_{[b|a]} \quad (8)$$

3. (1) Show that $T^{ab} = T^{abc} \delta^d_c$ is a tensor.

Solution: So, this is very long to write out, but here goes, under

$$x^a \rightarrow x'^a$$

we have

$$T^{abc} \rightarrow T'^{abc} = A^a_{a'} A^b_{b'} A^c_{c'} T^{abc} \quad (9)$$

Thus

$$T^{abc} \rightarrow T'^{abc} = A^a_{a'} A^b_{b'} A^c_{c'} T^{abc} \quad (10)$$

where we have used

$$A^a_{a'} A^b_{b'} = \delta^a_{b'} \quad (11)$$

Alternatively, observe that $\delta^a_{b'}$ is a tensor, say $T^a_b = \delta^a_{b'}$ in some coordinate system, then under a coordinate transformation

$$T^a_b \rightarrow T'^{b'}_{a'} = A^a_{a'} A^b_{b'} T^a_b \quad (12)$$

Thus, the Kronecker delta is a tensor. The contraction and multiplication properties of tensors can now be invoked.

4. (1) Show that $T^{ab} = -T^{ba}$ in one coordinate system implies that $T^{a'b'} = -T^{b'a'}$ in another coordinate system.

Solution: Well

$$T^{a'b'} = A^a_{a'} A^b_{b'} T^{ab} \quad (13)$$

and hence

$$T^{a'b'} = A^a_{a'} A^b_{b'} T^{ab} \quad (14)$$

then changing the dummy indices and then using the antisymmetry of $T$ we have

$$T^{a'b'} = A^a_{a'} A^b_{b'} T^{ba} = -A^b_{p'} A^a_{a'} T^{ab} = -T^{b'a'} \quad (15)$$

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Solution: Well

\[ T^{\alpha\beta} = A^\alpha_{\beta} A^\beta_{\alpha} T^{ab} \]  

(18)

and hence

\[ T^{\dot{\alpha}\dot{\beta}} = A^\dot{\alpha}_{\dot{\beta}} A^\dot{\beta}_{\dot{\alpha}} T^{\dot{ab}} \]  

(19)

then changing the dummy indices and then using the antisymmetry of \( T \) we have

\[ T^{\alpha\beta} = A^\alpha_{\beta} A^\beta_{\alpha} T^{ab} = -A^\beta_{\alpha} A^\alpha_{\beta} T^{ab} = -T^{\beta\alpha} \]  

(20)

5. (3) Write \( \Delta f \) in two-dimensional polar coordinates using the torsion free metric connection.

Solution: So,

\[ \Delta f = D_\alpha D^\alpha f = D_\alpha \partial^\alpha f \]  

(21)

where we have used the fact that \( f \) is a scalar. Now, from the formula for the covariant derivative

\[ \Delta f = \partial_\alpha \partial^\alpha f + \Gamma_{\alpha\beta}^\gamma \partial_\gamma f \]  

(22)

Hence, we need to work out the connection coefficients with the up index equal to the first down index. Now, for polar coordinates

\[ ds^2 = dr^2 + r^2 d\theta^2 \]  

(23)

or

\[ [g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \]  

(24)

and

\[ [g^{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \]  

(25)

Using the fact that this is diagonal, we have

\[ \Gamma^r_\theta = \frac{1}{2} g_{rr,\theta} = 0 \]
\[ \Gamma^r_{\theta\theta} = \frac{1}{2} g_{r\theta,\theta} = 0 \]
\[ \Gamma^\theta_\theta = \frac{1}{2} g_{\theta\theta,\theta} = 0 \]
\[ \Gamma^\theta_{\theta\theta} = \frac{1}{2} g_{\theta\theta,\theta} = \frac{1}{r} \]  

(26)

So the only nonzero entry is \( \Gamma^\theta_\theta \), and so we conclude

\[ \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r} \frac{\partial f}{\partial \theta} \]  

(27)

6. (3) Show that torsion is a tensor.

Solution: So,

\[ \text{det} g_{\alpha\beta} = \text{det} (A^\alpha_{\beta} A^\beta_{\alpha} g_{\alpha\beta}) = (\text{det} A^\alpha_{\beta})^2 \text{det} g_{\alpha\beta} \]  

(28)

Solution: So, to recall, the transformation property of the connection coefficients is

\[ \Gamma_{\alpha\beta}^{\gamma}_c = A_{a}^\gamma \Gamma_{\alpha\beta}^{\gamma}_a - A_{b}^\gamma \Gamma_{\alpha\beta}^{\gamma}_b \]  

(29)

Now, torsion is the anti-symmetric part of the connection

\[ T_{\alpha\beta} = \frac{1}{2} (\Gamma_{\alpha\beta}^{\gamma}_c - \Gamma_{\beta\alpha}^{\gamma}_c) \]  

(30)

Using the transformation law above, this means that

\[ 2 T_{\alpha\beta} = A_{a}^\gamma \Gamma_{\alpha\beta}^{\gamma}_a - A_{b}^\gamma \Gamma_{\alpha\beta}^{\gamma}_b - A_{c}^\gamma \Gamma_{\alpha\beta}^{\gamma}_c + A_{c}^\gamma \Gamma_{\alpha\beta}^{\gamma}_c \]  

(31)

Now, expanding out the notation,

\[ \partial_\alpha A^\gamma_c = \frac{\partial^2 x^\gamma}{\partial x^\alpha \partial x^\rho} \]  

(32)

is symmetric in \( b \) and \( c \), this allows us to cancel the two non-tensor terms in the transformation law, proving the result.

7. (1) Find the transformation law for \( \text{det} g_{ab} \).

Solution: So

\[ \text{det} g_{\alpha\beta} = \text{det} (A^\alpha_{\beta} A^\beta_{\alpha} g_{\alpha\beta}) = (\text{det} A^\alpha_{\beta})^2 \text{det} g_{\alpha\beta} \]  

(33)

8. (3) Show that \( \nabla_a g^{bc} = 0 \) for a torsion free metric connection.

Solution: First, from the Leibnitz rule

\[ \nabla_a (g^{bc} g_{bd}) = \nabla_a \delta^c_d \]  

(34)
and \( \nabla_a g_{bd} = 0 \) for a metric connection. Furthermore, since \( g^{bc} \) is defined as the inverse of \( g_{bd} \) we must be assuming that the metric is invertible and so, we need only to show \( \nabla_a \delta^b_0 = 0 \). From the action of the covariant derivative on a \((1,1)\) tensor we know

\[
\nabla_a \delta^b_0 = \partial_a \delta^b_0 + \Gamma^e_{a0} \delta^b_0 - \Gamma^e_{ab} \delta^e_0 \tag{35}
\]

but, the first term is zero since the delta is constant and the other two terms cancel.