

2 November 2004

Q1. (4) By explicit calculation show that the torsion free metric connection is a connection that is it obeys the coordinate transformation rule for a connection.

Firstly, the torsion free metric connection is given by

$$\Gamma_{bc}^a = \frac{1}{2}g^{ad}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}). \quad (1)$$

Note in the following calculation $g_{ab} = g_{ba}$.

Now $x^a \rightarrow x^{a'}$ then

$$g_{dc} \rightarrow g_{d'c'} = A_{d'}^d A_{c'}^c g_{dc} \quad (2)$$

so that

$$\begin{aligned} \partial_b g_{dc} \rightarrow \partial_{b'} g_{d'c'} &= \partial_{b'} (A_{d'}^d A_{c'}^c g_{dc}) \\ &= A_{b'}^b \partial_b (A_{d'}^d A_{c'}^c g_{dc}) \\ &= A_{b'}^b [(\partial_b A_{d'}^d) A_{c'}^c g_{dc} + A_{d'}^d \partial_b (A_{c'}^c g_{dc})] \\ &= A_{b'}^b \{(\partial_b A_{d'}^d) A_{c'}^c g_{dc} + A_{d'}^d [A_{c'}^c (\partial_b g_{dc}) + (\partial_b A_{c'}^c) g_{dc}]\} \\ &= A_{b'}^b \frac{\partial^2 x^d}{\partial x^b \partial x^{d'}} A_{c'}^c g_{dc} + A_{b'}^b A_{d'}^d A_{c'}^c \partial_b g_{dc} + A_{b'}^b A_{d'}^d \frac{\partial^2 x^c}{\partial x^b \partial x^{c'}} g_{dc} \end{aligned} \quad (3)$$

or

$$\partial_{b'} g_{d'c'} = \frac{\partial^2 x^d}{\partial x^{b'} \partial x^{d'}} A_{c'}^c g_{dc} + A_{b'}^b A_{d'}^d A_{c'}^c \partial_b g_{dc} + A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{dc} \quad (4)$$

By cyclic permutations of indices d', c', b' and d, c, b we get

$$\partial_{d'} g_{c'b'} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} A_{b'}^b g_{cb} + A_{d'}^d A_{c'}^c A_{b'}^b \partial_d g_{cb} + A_{c'}^c \frac{\partial^2 x^b}{\partial x^{d'} \partial x^{b'}} g_{cb} \quad (5)$$

and

$$\partial_{c'} g_{b'd'} = \frac{\partial^2 x^b}{\partial x^{c'} \partial x^{b'}} A_{d'}^d g_{bd} + A_{c'}^c A_{b'}^b A_{d'}^d \partial_c g_{bd} + A_{b'}^b \frac{\partial^2 x^d}{\partial x^{c'} \partial x^{d'}} g_{bd} \quad (6)$$

Now subtracting (4) from the sum of (3) and (5) and multiplying by $\frac{1}{2}$, we obtain

$$\begin{aligned} \text{LHS} &= \frac{1}{2}(\partial_{b'} g_{d'c'} + \partial_{c'} g_{b'd'} - \partial_{d'} g_{c'b'}) \\ &= \{c'b', d'\} \\ &= \{b'c', d'\}. \end{aligned}$$

¹Conor Houghton, houghton@maths.tcd.ie, see also <http://www.maths.tcd.ie/~houghton/442.html>
Thanks to Niall Dooley for help with this solution sheet

and

$$\begin{aligned} \text{RHS} &= \frac{\partial^2 x^d}{\partial x^{b'} \partial x^{d'}} A_{c'}^c g_{dc} + A_{b'}^b A_{d'}^d A_{c'}^c \partial_b g_{dc} + A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{dc} \\ &+ \frac{\partial^2 x^b}{\partial x^{c'} \partial x^{b'}} A_{d'}^d g_{bd} + A_{c'}^c A_{b'}^b A_{d'}^d \partial_c g_{bd} + A_{b'}^b \frac{\partial^2 x^d}{\partial x^{c'} \partial x^{d'}} g_{bd} \\ &- \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} A_{b'}^b g_{cb} - A_{d'}^d A_{c'}^c A_{b'}^b \partial_d g_{cb} - A_{c'}^c \frac{\partial^2 x^b}{\partial x^{d'} \partial x^{b'}} g_{cb} \\ &= A_{d'}^d A_{c'}^c A_{b'}^b (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{cb}) + (A_{c'}^c \frac{\partial^2 x^b}{\partial x^{b'} \partial x^{d'}} g_{bc})_{d \rightarrow b} \\ &+ (A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{bc})_{d \rightarrow b} + (A_{b'}^b \frac{\partial^2 x^c}{\partial x^{c'} \partial x^{b'}} g_{cb})_{d \rightarrow b, b \rightarrow c} \\ &+ (A_{b'}^b \frac{\partial^2 x^c}{\partial x^{c'} \partial x^{d'}} g_{bc})_{d \rightarrow b} - A_{b'}^b \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} g_{cb} - A_{c'}^c \frac{\partial^2 x^b}{\partial x^{d'} \partial x^{b'}} g_{cb} \\ &= A_{d'}^d A_{b'}^b A_{c'}^c \{cb, d\} + A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{bc} \\ &= A_{d'}^d A_{b'}^b A_{c'}^c \{bc, d\} + A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{bc} \end{aligned}$$

where for example $(A_{c'}^c \frac{\partial^2 x^b}{\partial x^{b'} \partial x^{d'}} g_{bc})_{d \rightarrow b}$ is $A_{c'}^c \frac{\partial^2 x^d}{\partial x^{b'} \partial x^{d'}} g_{dc}$ with the dummy index d changed to b .
Now equating, LHS = RHS get

$$\{b'c', d'\} = A_{b'}^b A_{c'}^c A_{d'}^d \{bc, d\} + A_{d'}^d \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} g_{bc}$$

$$\begin{aligned} \Rightarrow \Gamma_{b'c'}^{a'} &= g^{a'd'} \{b'c', d'\} \\ &= A_{b'}^b A_{c'}^c A_{d'}^d g^{a'd'} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_{d'}^d g^{a'd'} g_{bc} \\ &= A_{b'}^b A_{c'}^c A_{d'}^d A_a^{a'} A_e^{d'} g^{ae} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_{d'}^d A_a^{a'} A_e^{d'} g^{ae} g_{bc} \\ &= A_{b'}^b A_{c'}^c A_a^{a'} \delta_e^{d'} g^{ae} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_a^{a'} \delta_e^b g^{ae} g_{bc} \\ &= A_{b'}^b A_{c'}^c A_a^{a'} g^{ad} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_a^{a'} g^{ab} g_{bc} \\ &= A_{b'}^b A_{c'}^c A_a^{a'} \Gamma_{bc}^a + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_a^{a'} \delta_c^a \\ &= A_{b'}^b A_{c'}^c A_a^{a'} \Gamma_{bc}^a + \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} A_c^{a'} \\ &= A_{b'}^b A_{c'}^c A_a^{a'} \Gamma_{bc}^a + \frac{\partial^2 x^a}{\partial x^{b'} \partial x^{c'}} A_a^{a'}. \end{aligned}$$

on changing $c \rightarrow a$ in the 2nd term.

This is a nice way to write the transformation, and is equivalent to the form used derived in lectures:

$$\Gamma_{b'c'}^{a'} = A_{b'}^b A_{c'}^c A_a^{a'} \Gamma_{bc}^a - A_{b'}^b A_{c'}^c \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c}. \quad (7)$$

To see these are the same, note that

$$A_{c'}^c \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = -A_c^{a'} \frac{\partial^2 x^c}{\partial x^b \partial x^c} \quad (8)$$

because

$$A_{c'}^c \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = \frac{\partial x^c}{\partial x^{c'}} \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} \quad (9)$$

and, by the product rule,

$$\frac{\partial x^c}{\partial x^{c'}} \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = \frac{\partial}{\partial x^b} \left(\frac{\partial x^c}{\partial x^{c'}} \frac{\partial x^{a'}}{\partial x^c} \right) - \frac{\partial^2 x^c}{\partial x^b \partial x^{c'}} \frac{\partial x^{a'}}{\partial x^c} \quad (10)$$

Now, the first term on the right hand side is zero because it involves differentiating the Kroneker delta. Next, by the chain rule

$$\frac{\partial^2 x^c}{\partial x^b \partial x^{c'}} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} \frac{\partial x^{d'}}{\partial x^b} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} A_b^{d'} \quad (11)$$

Putting all this together

$$\begin{aligned} -A_{b'}^b A_{c'}^c \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} &= A_{b'}^b A_{c'}^c A_b^{d'} \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} \\ &= \delta_{b'}^{d'} A_c^{a'} \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} = A_c^{a'} \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} \end{aligned} \quad (12)$$

and c is a dummy index.