442 Tutorial Sheet 1b[1]

2 November 2004

Q1. (4) By explicit calculation show that the torsion free metric connection is a connection that is it obeys the coordinate transformation rule for a connection.

Firstly, the torsion free metric connection is given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_a g_{dc} + \partial_c g_{db} - \partial_d g_{bc}). \tag{1}$$

Note in the following calculation $g_{ab} = g_{ba}$.
Now $x^a \to x^{a'}$ then

$$g_{dc} \to g_{d'c'} = A^d_{d'} A^c_{c'} g_{dc} \tag{2}$$

so that

$$\begin{aligned}
\partial_b g_{dc} \to \partial_{b'} g_{d'c'} &= \partial_{b'}(A^d_{d'} A^c_{c'} g_{dc}) \qquad (3)\\
&= A^b_{b'}\partial_b(A^d_{d'} A^c_{c'} g_{dc})\\
&= A^b_{b'}[(\partial_b A^d_{d'})A^c_{c'} g_{dc} + A^d_{d'}\partial_b(A^c_{c'} g_{dc})]\\
&= A^b_{b'}\{(\partial_b A^d_{d'})A^c_{c'} g_{dc} + A^d_{d'}[A^c_{c'}(\partial_b g_{dc}) + (\partial_b A^c_{c'})g_{dc}]\}\\
&= A^b_{b'}\frac{\partial^2 x^d}{\partial x^b \partial x^{d'}}A^c_{c'} g_{dc} + A^b_{b'}A^d_{d'}A^c_{c'}\partial_b g_{dc} + A^b_{b'}A^d_{d'}\frac{\partial^2 x^c}{\partial x^b \partial x^{c'}}g_{dc}
\end{aligned}$$

or

$$\partial_{b'} g_{d'c'} = \frac{\partial^2 x^d}{\partial x^{b'} \partial x^{d'}}A^c_{c'} g_{dc} + A^b_{b'}A^d_{d'}A^c_{c'}\partial_b g_{dc} + A^d_{d'}\frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}}g_{dc} \tag{4}$$

By cyclic permutations of indices d', c', b' and d, c, b we get

$$\partial_{d'} g_{c'b'} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}}A^b_{b'} g_{cb} + A^d_{d'}A^c_{c'}A^b_{b'}\partial_d g_{cb} + A^c_{c'}\frac{\partial^2 x^b}{\partial x^{d'} \partial x^{b'}}g_{cb} \tag{5}$$

and

$$\partial_{c'} g_{b'd'} = \frac{\partial^2 x^b}{\partial x^{c'} \partial x^{b'}}A^d_{d'} g_{bd} + A^c_{c'}A^b_{b'}A^d_{d'}\partial_c g_{bd} + A^b_{b'}\frac{\partial^2 x^d}{\partial x^{c'} \partial x^{d'}}g_{bd} \tag{6}$$

Now subtracting (4) from the sum of (3) and (5) and multipling by $\frac{1}{2}$, we obtain

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}(\partial_{b'} g_{d'c'} + \partial_{c'} g_{b'd'} - \partial_{d'} g_{c'b'})\\
&= \{c'b', d'\}\\
&= \{b'c', d'\}.
\end{aligned}$$

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and

$$
\begin{aligned}
\text{RHS} &= \frac{\partial^2 x^d}{\partial x^{b'}\partial x^{d'}} A^c_{c'} g_{dc} + A^b_{b'} A^d_{d'} A^c_{c'} \partial_b g_{dc} + A^d_{d'} \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} g_{dc} \\
&+ \frac{\partial^2 x^b}{\partial x^{c'}\partial x^{b'}} A^d_{d'} g_{bd} + A^c_{c'} A^b_{b'} A^d_{d'} \partial_c g_{bd} + A^b_{b'} \frac{\partial^2 x^d}{\partial x^{c'}\partial x^{d'}} g_{bd} \\
&- \frac{\partial^2 x^c}{\partial x^{d'}\partial x^{c'}} A^b_{b'} g_{cb} - A^d_{d'} A^c_{c'} A^b_{b'} \partial_d g_{cb} - A^c_{c'} \frac{\partial^2 x^b}{\partial x^{d'}\partial x^{b'}} g_{cb} \\
&= A^d_{d'} A^c_{c'} A^b_{b'} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{cb}) + (A^c_{c'} \frac{\partial^2 x^b}{\partial x^{b'}\partial x^{d'}} g_{bc})_{d\to b} \\
&+ (A^b_{d'} \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} g_{bc})_{d\to b} + (A^b_{d'} \frac{\partial^2 x^c}{\partial x^{c'}\partial x^{b'}} g_{cb})_{d\to b, b\to c} \\
&+ (A^b_{b'} \frac{\partial^2 x^c}{\partial x^{c'}\partial x^{d'}} g_{bc})_{d\to b} - A^b_{b'} \frac{\partial^2 x^c}{\partial x^{d'}\partial x^{c'}} g_{cb} - A^c_{c'} \frac{\partial^2 x^b}{\partial x^{d'}\partial x^{b'}} g_{cb} \\
&= A^d_{d'} A^b_{b'} A^c_{c'} \{cb, d\} + A^b_{d'} \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} g_{bc} \\
&= A^d_{d'} A^b_{b'} A^c_{c'} \{bc, d\} + A^b_{d'} \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} g_{bc}
\end{aligned}
$$

where for example $(A^c_{c'} \frac{\partial^2 x^b}{\partial x^{b'}\partial x^{d'}} g_{bc})_{d\to b}$ is $A^c_{c'} \frac{\partial^2 x^d}{\partial x^{b'}\partial x^{d'}} g_{dc}$ with the dummy index d changed to b. Now equating, LHS = RHS get

$$
\{b'c', d'\} = A^b_{b'} A^c_{c'} A^d_{d'} \{bc, d\} + A^b_{d'} \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} g_{bc}
$$

$$
\begin{aligned}
\Rightarrow \Gamma^{a'}_{b'c'} &= g^{a'd'} \{b'c', d'\} \\
&= A^b_{b'} A^c_{c'} A^d_{d'} g^{a'd'} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^b_{d'} g^{a'd'} g_{bc} \\
&= A^b_{b'} A^c_{c'} A^d_{d'} A^{a'}_a A^{d'}_e g^{ae} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^b_{d'} A^{a'}_a A^{d'}_e g^{ae} g_{bc} \\
&= A^b_{b'} A^c_{c'} A^{a'}_a \delta^d_e g^{ae} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^{a'}_a \delta^b_e g^{ae} g_{bc} \\
&= A^b_{b'} A^c_{c'} A^{a'}_a g^{ad} \{bc, d\} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^{a'}_a g^{ab} g_{bc} \\
&= A^b_{b'} A^c_{c'} A^{a'}_a \Gamma^a_{bc} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^{a'}_a \delta^a_c \\
&= A^b_{b'} A^c_{c'} A^{a'}_a \Gamma^a_{bc} + \frac{\partial^2 x^c}{\partial x^{b'}\partial x^{c'}} A^{a'}_c \\
&= A^b_{b'} A^c_{c'} A^{a'}_a \Gamma^a_{bc} + \frac{\partial^2 x^a}{\partial x^{b'}\partial x^{c'}} A^{a'}_a .
\end{aligned}
$$

on changing $c \to a$ in the 2nd term.

This is a nice way to write the transformation, and is equivelent to the form used derived in lectures:

$$\Gamma^{a'}_{b'c'} = A^b_{b'}A^c_{c'}A^{a'}_a\Gamma^a_{bc} - A^b_{b'}A^c_{c'}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c}. \tag{7}$$

To see these are the same, note that

$$A^c_{c'}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = -A^{a'}_c \frac{\partial^2 x^c}{\partial x^b \partial x^c} \tag{8}$$

because

$$A^c_{c'}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = \frac{\partial x^c}{\partial x^{c'}}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} \tag{9}$$

and, by the product rule,

$$\frac{\partial x^c}{\partial x^{c'}}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} = \frac{\partial}{\partial x^b}(\frac{\partial x^c}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^c}) - \frac{\partial^2 x^c}{\partial x^b \partial x^{c'}}\frac{\partial x^{a'}}{\partial x^c} \tag{10}$$

Now, the first term on the right hand side is zero because it involves differenciating the Kroneker delta. Next, by the chain rule

$$\frac{\partial^2 x^c}{\partial x^b \partial x^{c'}} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}}\frac{\partial x^{d'}}{\partial x^b} = \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}}A^{d'}_b \tag{11}$$

Putting all this together

$$\begin{aligned} -A^b_{b'}A^c_{c'}\frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} &= A^b_{b'}A^{a'}_c A^{d'}_b \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} \\ &= \delta^{d'}_{b'}A^{a'}_c \frac{\partial^2 x^c}{\partial x^{d'} \partial x^{c'}} = A^{a'}_c \frac{\partial^2 x^c}{\partial x^{b'} \partial x^{c'}} \end{aligned} \tag{12}$$

and c is a dummy index.