442 Tutorial Sheet 1 Solutions[1]

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1. (1) Show that $T^{ab}S_b$ is a tensor, where T and S are tensors.

   Solution: So, since they are tensors, if

$$x^a \longrightarrow x^{a'} \tag{1}$$

then

$$\begin{aligned} T^{ab} &\longrightarrow T^{a'b'} = A^{a'}_a A^{b'}_b T^{ab} \\ S_a &\longrightarrow S_{a'} = A^a_{a'} S_a \end{aligned} \tag{2}$$

hence

$$T^{ab}S_b \to T^{a'b'}S'_b = A^{a'}_a A^{b'}_b T^{ab} A^c_{b'} S_c \tag{3}$$

but we know that

$$A^{b'}_b A^c_{b'} = \delta^c_b \tag{4}$$

so

$$T^{ab}S_b \to T^{a'b'}S'_b = A^{a'}_a T^{ab} S_b \tag{5}$$

as required.

2. (1) Show that $T^{(ab)}V_{[a|c|b]}$ vanishes, where T and V are tensors.

   Solution: Well, remember that

$$V_{[a|b|c]} = \frac{1}{2}(V_{abc} - V_{cba}) \tag{6}$$

and so $V_{[a|b|c]} = -V_{[c|b|a]}$. Furthermore

$$T^{(ab)} = \frac{1}{2}(T^{ab} + T^{ba}) \tag{7}$$

and $T^{(ab)} = T^{(ba)}$. Thus

$$T^{(ab)}V_{[a|c|b]} = -T^{(ab)}V_{[b|c|a]} \tag{8}$$

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and changing dummy indexes and then using the symmetry of $T^{(ab)}$

$$T^{(ab)}V_{[a|c|b]} = -T^{(ba)}V_{[a|c|b]} = -T^{(ab)}V_{[a|c|b]} \tag{9}$$

3. (1) Show that $T^{ab}{}_{e} = T^{abc}{}_{de}\delta^{d}_{c}$ is a tensor.

Solution: So, this is very long to write out, but here goes, under

$$x^{a} \rightarrow x^{a'} \tag{10}$$

we have

$$T^{abc}{}_{de} \rightarrow T^{a'b'c'}{}_{d'e'} = A^{a'}_{a}A^{b'}_{b}A^{c'}_{c}A^{d}_{d'}A^{e}_{e'}T^{abc}{}_{de} \tag{11}$$

Thus

$$T^{abd}{}_{de} \rightarrow T^{a'b'd'}{}_{d'e'} = A^{a'}_{a}A^{b'}_{b}A^{d'}_{c}A^{d}_{d'}A^{e}_{e'}T^{abc}{}_{de} = A^{a'}_{a}A^{b'}_{b}A^{e}_{e'}T^{abd}{}_{de} \tag{12}$$

where we have used

$$A^{d'}_{c}A^{d}_{d'} = \delta^{d}_{c} \tag{13}$$

Alternatively, observer that δ^{b}_{a} is a tensor, say $T^{b}_{a} = \delta^{b}_{a}$ in some coördinate system, then, under a coördinate transformation

$$T^{b}_{a} \mapsto T^{b'}_{a'} = A^{a}_{a'}A^{b'}_{b}T^{b}_{a} = A^{a}_{a'}A^{b'}_{b}\delta^{b}_{a} = A^{a}_{a'}A^{b'}_{a} = \delta^{b'}_{a'} \tag{14}$$

Thus, the Kronecker delta is a tensor. The contraction and multiplication properties of tensors can now be invoked.

4. (1) Show that $T^{ab} = -T^{ba}$ in one coördinate system implies that $T^{a'b'} = -T^{b'a'}$ in another coördinate system.

Solution: Well

$$T^{a'b'} = A^{a'}_{a}A^{b'}_{b}T^{ab} \tag{15}$$

and hence

$$T^{b'a'} = A^{b'}_{a}A^{a'}_{b}T^{ab} \tag{16}$$

then changin the dummy indices and then using the antisymmetry of T we have

$$T^{b'a'} = A^{b'}_{b}A^{a'}_{a}T^{ba} = -A^{b'}_{b}A^{a'}_{a}T^{ab} = -T^{a'b'} \tag{17}$$

Solution: Well

$$T^{a'b'} = A^{a'}_a A^{b'}_b T^{ab} \tag{18}$$

and hence

$$T^{b'a'} = A^{b'}_a A^{a'}_b T^{ab} \tag{19}$$

then changin the dummy indices and then using the antisymmetry of T we have

$$T^{b'a'} = A^{b'}_b A^{a'}_a T^{ba} = -A^{b'}_b A^{a'}_a T^{ab} = -T^{a'b'} \tag{20}$$

5. (3) Write $\triangle f$ in two-dimensional polar coordinates using the torsion free metric connection..

Solution: So,

$$\triangle f = D_a D^a f = D_a \partial^a f \tag{21}$$

where we have used the fact that f is a scalar. Now, from the formula for the covariant derivative

$$\triangle f = \partial_a \partial^a f + \Gamma^a_{ab} \partial^b f \tag{22}$$

Hence, we need to work out the connection coefficients with the up index equal to the first down index. Now, for polar coördinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{23}$$

or

$$[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{24}$$

and

$$[g^{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{25}$$

Using the fact that this is diagonal, we have

$$\begin{aligned} \Gamma^r_{rr} &= \frac{1}{2} g_{rr,r} = 0 \\ \Gamma^r_{r\theta} &= \frac{1}{2} g_{rr,\theta} = 0 \\ \Gamma^\theta_{\theta\theta} &= \frac{1}{2r^2} g_{\theta\theta,\theta} = 0 \\ \Gamma^\theta_{r\theta} &= \frac{1}{2r^2} g_{\theta\theta,r} = \frac{1}{r} \end{aligned} \tag{26}$$

So the only nonzero entry is $\Gamma^\theta_{r\theta}$ and so we conclude

$$\triangle f = \frac{\partial^2}{\partial r^2} f + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} f + \frac{1}{r} \frac{\partial}{\partial r} f \tag{27}$$

6. (3) Show that torsion is a tensor.

Solution: So,

$$\det g_{a'b'} = \det\left(A^a_{a'}A^b_{b'}g_{a'b'}\right) = (\det A^a_{a'})^2 \det g_{ab} \tag{28}$$

Solution: So, to recall, the transformation property of the connection coefficients is

$$\Gamma^a_{bc} \to \Gamma^{a'}_{b'c'} = A^b_{b'}A^c_{c'}A^{a'}_a \Gamma^a_{bc} - A^b_{b'}A^c_{c'}(\partial_b A^{a'}_c) \tag{29}$$

Now, torsion is the anti-symmetric part of the connection

$$T^a_{bc} = \frac{1}{2}\left(\Gamma^a_{bc} - \Gamma^a_{cb}\right) \tag{30}$$

Using the tranformation law above, this means that

$$\begin{aligned} 2T^{a'}_{b'c'} &= A^b_{b'}A^c_{c'}A^{a'}_a \Gamma^a_{bc} - A^b_{b'}A^c_{c'}(\partial_b A^{a'}_c) - A^b_{c'}A^c_{b'}A^{a'}_a \Gamma^a_{bc} + A^b_{c'}A^c_{b'}(\partial_b A^{a'}_c) \\ &= A^b_{b'}A^c_{c'}A^{a'}_a 2T^a_{bc} + A^b_{c'}A^c_{b'}(\partial_b A^{a'}_c) - A^b_{b'}A^c_{c'}(\partial_b A^{a'}_c) \end{aligned} \tag{31}$$

Now, expanding out the notation,

$$\partial_b A^{a'}_c = \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} \tag{32}$$

is symmetric in b and c, this allows us to cancel the two non-tensor terms in the transformation law, proving the result.

7. (1) Find the transformation law for detg_{ab}.

Solution: So

$$\det g_{a'b'} = \det\left(A^a_{a'}A^b_{b'}g_{a'b'}\right) = (\det A^a_{a'})^2 \det g_{ab} \tag{33}$$

8. (3) Show that $\nabla_a g^{bc} = 0$ for a torsion free metric connection.

Solution: First, from the Leibnitz rule

$$\nabla_a \left(g^{bc} g_{cd}\right) = \nabla_a \delta^b_d$$

$$= \left(\nabla_a g^{bc}\right) g_{cd} + g^{bc}\nabla_a g_{cd} \tag{34}$$

and $\nabla_a g_{cd} = 0$ for a metric connection. Furthermore, since g^{bc} is defined as the inverse of g_{cd} we must be assuming that the metric is invertible and so, we need only to show $\nabla_a \delta^b_d = 0$. From the action of the covariant derivative on a (1,1) tensor we know

$$\nabla_a \delta^b_d = \partial_a \delta^b_d + \Gamma^d_{ae}\delta^e_b - \Gamma^e_{ab}\delta^d_e \tag{35}$$

but, the first term is zero since the delta is constant and the other two terms cancel.