This is a sample paper intended to give an indication as to the style of the paper, however, I haven’t made up new questions so they are all off the Problem Sheets, in the exam this won’t be the case, in the exam roughly half the non-bookwork material will be original and half will have been in a problem sheet, perhaps with small differences.

1. (a) Define geodesic coordinates and writing the Riemann tensor in terms of these coordinates show $R_{abcd} = R_{cdab}$. [BOOK WORK]

(b) Find the scalar curvature on a two-dimensional hyperboloid:

$$x^2 + y^2 - t^2 = -r^2$$

embedded in the Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2$$

2. Define a geodesic and prove that the geodesic gives the shortest path between two points. [BOOK WORK] Find the time-like geodesics for the metric

$$ds^2 = \frac{1}{t^2} (-dt^2 + dx^2)$$

You might want to use the integral

$$\int \frac{dt}{t\sqrt{1 + C^2 t^2}} = \frac{1}{2} \log \left( \frac{\sqrt{1 + C^2 t^2} - 1}{\sqrt{1 + C^2 t^2} + 1} \right)$$

3. In Einstein’s theory of gravity a test particle moves on a geodesic in a space-time whose curvature is determined by the Einstein equation. In Newton’s theory of gravity a test particle has an acceleration according to Newton’s law with the force determined by the Poisson equation. Show that in the non-relativistic weak-field approximation the Einsteinian gravity for a static space-time is approximated by Newtonian gravity. [BOOK WORK]

4. Let

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where, by a choice of coordinates, $h_{\mu\nu}$ satisfies the harmonic gauge condition:

$$\partial_{\mu} h^\mu_{\nu} = \frac{1}{2} \partial_{\nu} h$$
show that the Einstein equations reduce to

$$\frac{1}{2} \partial^2 h_{\mu\nu} + \frac{1}{4} \eta_{\mu\nu} \partial^2 h = 8\pi T_{\mu\nu}$$

(7)

where, as usual, $h = h^\mu_\mu$ is the trace. By taking the trace of both sides and solving for $\partial^2 h$, show that this can be written as

$$\partial^2 h_{\mu\nu} = -16\pi \left( T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right)$$

(8)

where $T = \eta^{\mu\nu} T_{\mu\nu}$.

5. Calculate the energy-momentum tensor

$$T_{\mu\nu} = F^\rho_{\mu\nu} F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho}$$

for the Maxwell field:

$$S = -\frac{1}{4} \int F^{\mu\nu} F_{\mu\nu} \sqrt{g} d^4x$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

where you may quote the variation $\delta g = gh/2$ for $\delta g_{\mu\nu} = h_{\mu\nu}$ and $h = \eta^{\mu\nu} h_{\mu\nu}$. Verify that this expression satisfies the identity $\nabla^\mu T_{\mu\nu} = 0$.

6. Show that a universe with $\Omega_0 = 1 - \epsilon$ for small positive $\epsilon$ has age

$$t_0 = \frac{2}{3H_0} \left( 1 + \frac{1}{5} \epsilon \right) + O(\epsilon^2).$$

You might find the expansion

$$\sinh^{-1} x = x - \frac{1}{6} x^3 + \frac{3}{40} x^5$$

useful, you might also benefit from being reminded that the usual way of integrating

$$\int \frac{dx}{\sqrt{\frac{1}{2} x + 1}}$$

is to substitute $x = A \sinh^2 \theta$ and then use

$$\int d\theta \sinh^2 \theta = \frac{1}{2} \sinh \theta \cosh \theta - \frac{1}{2} \theta$$

[BOOK WORK]
7. The equation of state is often written in *adiabatic form*

\[ p = (\gamma - 1)\rho \]

where \( p \) is pressure and \( \rho \) is density and \( 0 \leq \gamma \leq 2 \) is the *adiabatic index* with \( \gamma = 1 \) for dust and \( \gamma = 4/3 \) for radiation.

(a) Calculate \( \rho(a) \) for general \( \gamma \). Find the age of the universe for \( k = 0 \).

(b) Find \( \gamma \) so that the expansion rate is constant. With this value of \( \gamma \) find \( a(t) \) for \( k = 1 \) and \( k = -1 \).

(c) In the same notation, show

\[ \dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega) \]

Define the *logarithmic scale factor* \( s = \log a \) and write an equation for \( d\Omega/ds \).

(d) Comment on the flatness problem.

8. Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods: (a) \( 0 < t < t_3 \) radiation only; (b) \( t_3 < t < t_2 \) vacuum energy dominates with an effective cosmological constant \( \Lambda = 3t_3^2/4 \); (c) \( t_2 < t < t_1 \) a period of radiation domination; (d) \( t_1 < t < t_0 \) matter domination.

(a) Show that in (c) \( \rho(t) = \rho_r(t) = 3\pi t^2/32 \) and in (d) \( \rho(t) = \rho_m(t) = \pi t^2/6 \). The functions \( \rho_r \) and \( \rho_m \) are introduced for later convenience.

(b) Give simple analytic formulas for \( a(t) \) which are approximately true in these four epochs.

(c) Shat that during the inflationary epoch the universe expands by a factor

\[ \frac{a(t_2)}{a(t_3)} = \exp \left( \frac{t_2 - t_3}{2t_3} \right) \tag{9} \]

(d) In the notation introduced earlier, show

\[ \frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16} \left( \frac{t_1}{t_0} \right)^{2/3} \tag{10} \]

(e) If \( t_3 = 10^{-35} \) seconds, \( t_2 = 10^{-32} \) seconds, \( t_1 = 10^4 \) years and \( t_0 = 10^{10} \) years, give a sketch of \( \log a \) against \( \log t \) marking any important epochs.

9. Kaluza-Klein question, will be added later.