442 Tutorial Sheet 4[1]

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1. (4) Find the Killing vectors for flat space $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$. [Write out Killing's equation in flat space, differenciate it once and then solve the resulting differential equation].

2. (3) This question concerns the weak field limit. Let

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

where, by a choice of coordinates, $h_{\mu\nu}$ satisfies the harmonic gauge condition:

$$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h \tag{2}$$

show that the Einstein equations reduce to

$$-\frac{1}{2}\partial^2 h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\partial^2 h = 8\pi T_{\mu\nu} \tag{3}$$

where, as usual, $h = h^\mu_\mu$ is the trace. By taking the trace of both sides and solving for $\partial^2 h$, show that this can be written as

$$\partial^2 h_{\mu\nu} = -16\pi\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \tag{4}$$

where $T = \eta^{\mu\nu}T_{\mu\nu}$. These are the linearized Einstein equations.

3. (4) Cosmic strings are long heavy string-like objects which are possibly formed in the Universe during certain cooling transitions. They were once thought to be resposible for structure formation. In Cartesian coordinates, the energy momentum tensor for a cosmic string aligned along the z-axis may be approximated by

$$T_{\mu\nu} = \mu\delta(x)\delta(y)\text{diag}(1, 0, 0, -1) \tag{5}$$

where μ is a small positive constant. Working to linear order in μ show, using the linearized equations above that the change to the Minkowski metric is given by

$$h_{11} = h_{22} = -8\mu\log\frac{r}{r_0} \tag{6}$$

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with all others perturbations zero. Here, $r = \sqrt{x^2 + y^2}$ and r_0 is a length-scale which cannot be determined here, physically it is related to the radius of validity of the thin string approximation. By a change of variable in r

$$\left(1 - 8\mu \log \frac{r}{r_0}\right) r^2 = (1 - 8\mu) r'^2 \tag{7}$$

and in the azimuthal angle ϕ, show that the metric can still be written in the cylindrical form

$$ds^2 = -dt^2 + dz^2 + dr'^2 + r'^2 d\phi'^2 \tag{8}$$

but, the new azimuthal angle ϕ' does not have period 2π.

4. (2) We have seen in the last question that a cosmic string causes a conical singularity, space is flat, but the azimuthal angle has period less than 2π. Argue that cosmic strings cause double images of distant objects.