442 Summer 2004 q5 outline solution.

5 Consider a flat dust universe with zero cosmological constant

(a) Solve the cosmological equations and derive the time evolution of the scale parameter $a(t)$.

(b) By considering light emitted at time t_e and received at the present time t_0 show that the distance to a star of red-shift z is given by

$$s = 3t_0 \left(1 - \frac{1}{\sqrt{1+z}}\right)$$

(c) Explain why a flat universe with zero cosmological constant containing a mixture of dust and radiation will eventually be dominated by the dust.

Solution: So with $p = 0$

$$\dot{\rho} + \frac{3\dot{a}}{a}\rho = 0 \tag{1}$$

hence

$$\frac{d}{dt}(a^3\rho) = 0 \tag{2}$$

or

$$\rho(t) = \rho(t_0)\left[\frac{a(t_0)}{a(t)}\right]^3 \tag{3}$$

Next, the Freiedmann equation for a flat universe has

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} \tag{4}$$

and the accelleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho \tag{5}$$

Eliminating ρ and using the trick of writing

$$\begin{aligned} v &= \dot{a} \\ \ddot{a} &= v\frac{dv}{da} \end{aligned} \tag{6}$$

leads to

$$v^2 + 2av = 0 \tag{7}$$

or

$$\frac{d}{da}(v^2 a) = 0 \tag{8}$$

and hence

$$v = \sqrt{\frac{c}{a}} \tag{9}$$

where c is a constant. Finally, substituting back for v

$$\frac{da}{dt} = \sqrt{\frac{c}{a}} \tag{10}$$

or

$$a = \lambda t^{2/3} \tag{11}$$

where λ is some constant.

From the definition of redshift

$$\frac{a_e}{a_0} = \left(\frac{t_e}{t_0}\right)^{2/3} = \frac{1}{1+z} \tag{12}$$

where $a_0 = a(t_0)$, $a_e = a(t_e)$ and t_e is the time the light was emitted. Along a null-geodesic $dt = ad\tilde{s}$ and the actual distance to the star is the distance in the fixed three-dimensional space multiplied by the current scale parameter:

$$s = a_0 \int d\tilde{s} = a_0 \int \frac{dt}{a} \tag{13}$$

Substituting back in for a we get

$$s = a_0 \int_{t_e}^{t_0} \frac{dt}{\lambda t^{3/2}} = 3\left(t_0 - t_e^{1/3} t_0^{2/3}\right) \tag{14}$$

where we have cancelled the λs by substituting in for a_0. Finally, we move stuff around

$$s = 3t_0\left(1 - t_e^{1/3} t_0^{-1/3}\right) = 3t_0\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{15}$$

Finally, the density of radiation goes like $1/a^4$ so it scales away faster than dust and hence, dust will always dominate in the end.