5 Consider a flat dust universe with zero cosmological constant

(a) Solve the cosmological equations and derive the time evolution of the scale parameter $a(t)$.

(b) By considering light emitted at time $t_e$ and received at the present time $t_0$ show that the distance to a star of red-shift $z$ is given by

$$s = 3t_0 \left(1 - \frac{1}{\sqrt{1 + z}}\right)$$

(c) Explain why a flat universe with zero cosmological constant containing a mixture of dust and radiation will eventually be dominated by the dust.

Solution: So with $p = 0$

$$\dot{\rho} + \frac{3\dot{a}}{a} \rho = 0$$

(1)

hence

$$\frac{d}{dt} (a^3 \rho) = 0$$

(2)

or

$$\rho(t) = \rho(t_0) \left[ \frac{a(t_0)}{a(t)} \right]^3$$

(3)

Next, the Freiedmann equation for a flat universe has

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi \rho}{3}$$

(4)

and the acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} \rho$$

(5)

Eliminating $\rho$ and using the trick of writing

$$v = \dot{a} \quad \ddot{a} = v \frac{dv}{da}$$

(6)

leads to

$$v^2 + 2av = 0$$

(7)

or

$$\frac{d}{da} (v^2 a) = 0$$

(8)
and hence
\[ v = \sqrt{\frac{c}{a}} \]  
(9)
where \( c \) is a constant. Finally, substituting back for \( v \)
\[ \frac{da}{dt} = \sqrt{\frac{c}{a}} \]  
(10)
or
\[ a = \lambda t^{2/3} \]  
(11)
where \( \lambda \) is some constant.

From the definition of redshift
\[ \frac{a_e}{a_0} = \left( \frac{t_e}{t_0} \right)^{2/3} = \frac{1}{1 + z} \]  
(12)
where \( a_0 = a(t_0), \ a_e = a(t_e) \) and \( t_e \) is the time the light was emitted. Along a
null-geodesic \( dt = ad\tilde{s} \) and the actual distance to the star is the distance in the fixed
three-dimensional space multiplied by the current scale parameter:
\[ s = a_0 \int d\tilde{s} = a_0 \int \frac{dt}{a} \]  
(13)
Substituting back in for \( a \) we get
\[ s = a_0 \int_{t_0}^{t_e} \frac{dt}{\lambda t^{3/2}} = 3 \left( t_0 - \frac{t_e^{1/3} t_0^{2/3}}{2} \right) \]  
(14)
where we have cancelled the \( \lambda s \) by substituting in for \( a_0 \). Finally, we move stuff
around
\[ s = 3t_0 \left( 1 - t_e^{1/3} t_0^{-1/3} \right) = 3t_0 \left( 1 - \frac{1}{\sqrt{1 + z}} \right) \]  
(15)
Finally, the density of radiation goes like \( 1/a^4 \) so it scales away faster than dust and
hence, dust will always dominate in the end.