442 Summer 2004 q2 (c) and (d) outline solution.

1. (a) Define an isometry.
   (b) Define the Killing vector and show that it satisfies
   $$\nabla_a k_b + \nabla_b k_a = 0.$$
   (c) Show how the Killing vector defines a constant along geodesics.
   (d) If k_a and l_a are Killing vectors, show that
   $$[k,l]_a = k^b \nabla_b l_a - l^b \nabla_b k_a$$
   is also a Killing vector. It might be useful to recall the symmetry properties of the Riemann tensor $R_{abcd} = -R_{bacd}$ and $R_{abcd} = R_{cdab}$.

Solution: So consider $C = k^a U_a$: along a geodesic we have

$$\begin{aligned} U^a \nabla_a C &= U^a \nabla_a (k^b U_b) \\ &= (U^a \nabla_a k^b) U_b + U^a \nabla_a U_b k^b \end{aligned} \tag{1}$$

Now, the second term is zero by the geodesic equation and the first term can be made equal to the geodesic equation by symmetrizing

$$(U^a \nabla_a k^b) U_b = U^a U^b (\nabla_a k_b) = \frac{1}{2} U^a U^b (\nabla_a k_b + \nabla_b k_a) \tag{2}$$

Next, consider substituting the commutator into the Killing equation

$$\begin{aligned} \nabla_a [k,l]_b + \nabla_b [k,l]_a &= \nabla_a (k^c \nabla_c l_b - l^c \nabla_c k_b) + \nabla_b (k^c \nabla_c l_a - l^c \nabla_c k_a) \\ &= \nabla_a k^c \nabla_c l_b + k^c \nabla_a \nabla_c l_b - \nabla_a l^c \nabla_c k_b - l^c \nabla_a \nabla_c k_b + (a \leftrightarrow b) \end{aligned}$$

Now, the big trick is to swap the summed index off the nabla onto the Killing vector using the Killing equation

$$\nabla_a [k,l]_b + \nabla_b [k,l]_a = -\nabla_a k^c \nabla_b l_c - k^c \nabla_a \nabla_c l_b + \nabla_a l^c \nabla_b k_c + l^c \nabla_a \nabla_c k_b + (a \leftrightarrow b) \tag{3}$$

we also swap the double nabla terms using the definition of the Riemann tensor

$$\begin{aligned} \nabla_a [k,l]_b + \nabla_b [k,l]_a = &-\nabla_a k^c \nabla_b l_c - k^c \nabla_c \nabla_a l_b - k^c R_{acbd} l^d + \\ &\nabla_a l^c \nabla_b k_c + l^c R_{acbd} k^d + l^c \nabla_c \nabla_a k_b + (a \leftrightarrow b) \end{aligned} \tag{4}$$

Now, add in the $(a \leftrightarrow b)$ part, the bits with the nabla's seperate cancel using the Killing equation, the double nabla terms are also zero by the Killing equation and we are left with

$$\begin{aligned} \nabla_a [k,l]_b + \nabla_b [k,l]_a &= -k^c R_{acbd} l^d + l^c R_{acbd} k^d - k^c R_{bcad} l^d + l^c R_{bcad} k^d \\ &= k^c l^d (R_{acbd} + R_{bcad} + R_{adbc} + R_{bdac}) \\ &= k^c l^d (-R_{acbd} + R_{adbc} - R_{adbc} + R_{acbd}) = 0 \end{aligned} \tag{5}$$