442 Summer 2004 q1 outline solution.

1. A paraboloid in three dimensional Eulclidean space

$$ds^2 = dx^2 + dy^2 + dz^2$$

is given by the equations

$$\begin{aligned} x &= u\cos\phi \\ y &= u\sin\phi \\ z &= \frac{u^2}{2} \end{aligned}$$

where $u \geq 0$ and $0 \leq \phi \leq 2\pi$.

(a) Show that the metric on the paraboloid is given by

$$ds^2 = (1+u^2)du^2 + u^2 d\phi^2$$

(b) Writing $x^1 = u$ and $x^2 = \phi$ find the Christoffel symbols for this metric.

(c) Solve the equation of parallel transport

$$U^a \nabla_a V^b = 0$$

where

$$U^a = \frac{dx^a}{dt}$$

for the curve $u = u_0$ where $u_0$ is a positive constant and with initial conditions $V^1 = 1$ and $V^2 = 0$. [Hint: the problem is simplified if you take $t = \phi$. The equation of parallel transport will give you two coupled equations for $U^1$ and $U^2$, differentiating the $dU^1/d\phi$ equation again allows you to decouple the $U^1$ equation.]

*Solution:* To get the metric do the usual chain rule on the coördinate change:

$$\begin{aligned} dx &= \cos\phi du - u\sin\phi d\phi \\ dy &= \sin\phi du + u\cos\phi d\phi \\ dz &= udu \end{aligned} \tag{1}$$

and then substitute into $dx^2 + dy^2 + dz^2$ and use the Pythagorous theorem.

Now for the connection coefficients you just use the formula and the inverse metric

$$[g^{ab}] = \begin{pmatrix} \frac{1}{1+u^2} & 0 \\ 0 & \frac{1}{u^2} \end{pmatrix} \tag{2}$$

This gives

$$\Gamma^1_{11} = \Gamma^1_{22} = -\frac{u}{1+u^2}$$

$$\Gamma^1_{12} = \Gamma^2_{21} \quad = \quad \frac{1}{u} \tag{3}$$

and all others zero. Note of course, that the two nonzero superscript one coefficients just happen to be the same, but for superscript two there is a symmetry. Now, in the parallel transport equation the curve has constant $u$ so it is given by $(u_0, \phi)$ and hence

$$(U^1, U^2) = (0, 1) \tag{4}$$

and the equation of parallel transport is

$$\frac{\partial}{\partial \phi} V^b + \Gamma^b_{\phi c} V^c = 0 \tag{5}$$

There is no $u$ derivative and, in fact, $u$ is constant on the curve, so we can assume that $V^b$ is constant in $u$. Putting in the values of the connection coefficients we get

$$\begin{aligned} \frac{\partial}{\partial \phi} V^1 - \frac{u}{1+u^2} V^2 &= 0 \\ \frac{\partial}{\partial \phi} V^2 + \frac{1}{u} V^1 &= 0 \end{aligned} \tag{6}$$

Diffenciating the second equation again and substituting in from the first gives

$$\frac{\partial^2}{\partial \phi^2} V^1 + \frac{1}{1+u^2} V^1 = 0 \tag{7}$$

so

$$V^1 = C_1 \cos f\phi + C_2 \sin f\phi \tag{8}$$

where

$$f = \frac{1}{\sqrt{1+u_0^2}} \tag{9}$$

where we have substituted for the value of $u$. Using the first equation again gives $V^2$

$$\begin{aligned} V^2 &= \frac{1+u^2}{u} \frac{\partial}{\partial \phi} V^1 \\ &= -\frac{\sqrt{1+u^2}}{u} \left( C_1 \sin f\phi + C_2 \cos -f\phi \right) \end{aligned} \tag{10}$$

Finally the initial conditions give $C_1 = 1$ and $C_2 = 0$.