442 Summer 2004

1. A paraboloid in three dimensional Eulclidean space

$$ds^2 = dx^2 + dy^2 + dz^2$$

is given by the equations

$$\begin{aligned} x &= u\cos\phi \\ y &= u\sin\phi \\ z &= \frac{u^2}{2} \end{aligned}$$

where $u \geq 0$ and $0 \leq \phi \leq 2\pi$.

   (a) Show that the metric on the paraboloid is given by

   $$ds^2 = (1+u^2)du^2 + u^2 d\phi^2$$

   (b) Writing $x^1 = u$ and $x^2 = \phi$ find the Christoffel symbols for this metric.

   (c) Solve the equation of parallel transport

   $$U^a \nabla_a V^b = 0$$

   where

   $$U^a = \frac{dx^a}{dt}$$

   for the curve $u = u_0$ where $u_0$ is a positive constant and with initial conditions $V^1 = 1$ and $V^2 = 0$. [Hint: the problem is simplified if you take $t = \phi$. The equation of parallel transport will give you two coupled equations for $U^1$ and $U^2$, differentiating the $dU^1/d\phi$ equation again allows you to decouple the $U^1$ equation.]

2. (a) Define an isometry.

   (b) Define the Killing vector and show that it satisfies

   $$\nabla_a k_b + \nabla_b k_a = 0.$$

   (c) Show how the Killing vector defines a constant along geodesics.

   (d) If $k_a$ and $l_a$ are Killing vectors, show that

   $$[k,l]_a = k^b \nabla_b l_a - l^b \nabla_b k_a$$

   is also a Killing vector. It might be useful to recall the symmetry properties of the Riemann tensor $R_{abcd} = -R_{bacd}$ and $R_{abcd} = R_{cdab}$.

3. In Einstein's theory of gravity a test particle moves on a geodesic in a space-time whose curvature is determined by the Einstein equation. In Newton's theory of gravity a test particle has an acceleration according to Newton's law with the force determined by the Poisson equation. Show that in the non-relativistic weak-field approximation the Einsteinian gravity for a static space-time is approximated by Newtonian gravity.

4. (a) Define normal coördinates and use them to prove the cyclic identity for the Riemann tensor

$$R_{abc}{}^{d} + R_{bac}{}^{d} + R_{cab}{}^{d} = 0$$

You do not need to prove that normal coördinates can always be found. It might be useful to recall

$$R_{abc^d} = R_{abc}{}^{d} = \Gamma^d_{ac,b} - \Gamma^d_{bc,a} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae}$$

(b) Calculate the energy-momentum tensor

$$T_{ab} = F_{ea}F^{e}{}_{b} - \frac{1}{4}g_{ab}F_{cd}F^{cd}$$

for the Maxwell field:

$$S = -\frac{1}{4}\int F^{ab}F_{ab}\sqrt{g}d^4x$$

where

$$F_{ab} = \nabla_a A_b - \nabla_b A_a$$

where you may quote the variation $\delta g = gh/2$ for $\delta g_{ab} = h_{ab}$ and $h = \eta^{ab}h_{ab}$.

(c) Verify that the energy momentum tensor for Maxwell fields satisfies

$$\nabla^a T_{ab} = 0$$

5. Consider a flat dust universe with zero cosmological constant

(a) Solve the cosmological equations and derive the time evolution of the scale parameter $a(t)$.

(b) By considering light emitted at time $t_e$ and received at the present time $t_0$ show that the distance to a star of red-shift $z$ is given by

$$s = 3t_0\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

(c) Explain why a flat universe with zero cosmological constant containing a mixture of dust and radiation will eventually be dominated by the dust.

6. Explain which problem prompted Einstein to introduce the cosmological constant, how the cosmological constant solves this problem and why this solution is not satisfactory. One advantage of a cosmological model with a cosmological constant is that it makes the universe older for given observational data. Demonstrate this for a flat dust universe.

7. The equation of state is often written in *adiabatic form*

$$p = (\gamma - 1)\rho$$

   where $p$ is pressure and $\rho$ is density and $0 \leq \gamma \leq 2$ is the *adiabatic index* with $\gamma = 1$ for dust and $\gamma = 4/3$ for radiation.

   (a) Calculate $\rho(a)$ for general $\gamma$. Find the age of the universe for $k = 0$.

   (b) Find $\gamma$ so that the expansion rate is constant. With this value of $\gamma$ find $a(t)$ for $k = 1$ and $k = -1$.

   (c) In the same notation, show

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega)$$

   Define the *logarithmic scale factor* $s = \log a$ and write an equation for $d\Omega/ds$.

   (d) Comment on the flatness problem.

8. The Universe undergoes a period of inflation driven by a scalar field with potential $V(\phi) = m^2\phi^2/2$. Write down the slow-roll equations and find the solutions for the scale factor $a$ and the field $\phi$ for initial conditions $a = a_i$ and $\phi = \phi_i$ at $t = 0$. For what range of $\phi$ values is this solution inflationary? What condition must be satisfied by the initial scalar field $\phi_i$ to ensure that an expansion by at least $10^{30}$ occurs.

9. Show that their is a coördinate choice so that the linearized vacuum Einstein equations are $\partial^2 h_{ab} = 0$ where $G_{ab} = \eta_{ab} + h_{ab}$. find the plane wave solutions to this equation and explain why there are only two polarizations. The transverse trace-free polarization has basis

$$e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   and

$$e_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   By taking the $e_+$ polarization, describe the physical effect of a gravitational plane wave.