General Relativity & Cosmology - Lecture 1

1 References

- d'Inverno, Ray - "Introducing Einstein's Relativity" - our main reference for GR
- Schutz, Bernard - "A First Course in General Relativity"

2 Prelude: Planck Units

In fundamental physics there are three dimensionfull constants; G, c, $\hbar$ (Newton's gravitational constant, the speed of light and Planck's constant, respectively).

G	-	strength of gravity, obtained from $F = G\frac{m_1 m_2}{r^2}$
	-	it is a weak force.
	:=	$6.673 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^2$
$[G]$	=	$\text{L}^3\text{M}^{-1}\text{T}^2$
c	:=	$2.997 \times 10^8\text{ms}^{-1}$
$[c]$	=	LT^{-1}
$\hbar$	:=	$1.054 \times 10^{-34}\text{Js}$
	-	sets the scale of quantum mechanical effects.
	-	is a quantum of work or action (energy x time).
	-	processes on this scale are quantum mechanical.
$[\hbar]$	=	ML^2T^{-2}

We define basic units of length, mass and time using these constants.

Planck mass, length and time are defined as

$$m_{pl} = \sqrt[2]{\frac{\hbar c}{G}} \qquad l_{pl} = \sqrt[2]{\frac{\hbar G}{c^3}} \qquad \tau_{pl} = \sqrt[2]{\frac{\hbar G}{c^5}} \tag{1}$$

So m_{pl} is a mass in Planck units:

$$[m_{pl}] = \left(\text{ML}^2\text{T}^{-1}\right)^{1/2}\left(\text{LT}^{-1}\right)^{1/2}\left(\text{L}^3\text{M}^{-1}\text{T}^{-2}\right)^{-1/2} = \text{M} \tag{2}$$

We refer to mass in m_{pl} rather than $x$kg.

$$l_{pl}^3 m_{pl}^{-1}\tau_{pl}^{-2} = \left(\frac{\hbar^{3/2}G^{3/2}}{c^{9/2}}\right)\left(\frac{G^{1/2}}{\hbar^{1/2}c^{1/2}}\right)\left(\frac{c^5}{\hbar G}\right) = G$$

$$\Rightarrow \quad G = 1\, l_{pl}^3 m_{pl}^{-1}\tau_{pl}^{-2}$$

In other words (and by further checking)

$$\left.\begin{array}{rcl} G &=& 1 \\ c &=& 1 \\ \hbar &=& 1 \end{array}\right\}\text{Planck Units}$$

For convenience we will use Planck units.

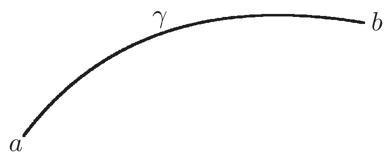
Before going on, notice the remarkable matching of the fundamental dimensional constants and the number of dimensions.

$$\left.\begin{array}{ccl} c & \longleftrightarrow & \text{special relativity} \\ \hbar & \longleftrightarrow & \text{quantum mechanics} \end{array}\right\} QFT$$
$$G \longleftrightarrow \text{general relativity}$$

(fitting QFT & GR together is hard - String Theory).
All of this leaves out "emergent behaviour" e.g. condensed matter physics, biology, chemistry.

3 Motivating Metrics

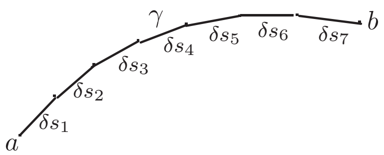
Consider two points a and b with a path γ between them:



The distance between a and b along γ is given by

$$s = \int_\gamma ds$$

But what exactly is "ds"? $\Rightarrow$ naïvely it is an infinitesimal increment along the path.
So the idea we have is that we approximate the path with a series of small increments and sum them



$$s \simeq \sum_{i=0}^{7} \delta s_i$$

As we increase the number of increments, this becomes more exact and $\int_\gamma ds$ is the limit

$$\sum_{i=0}^{N} \delta s_i \overset{N\to\infty}{\longrightarrow} \int ds$$

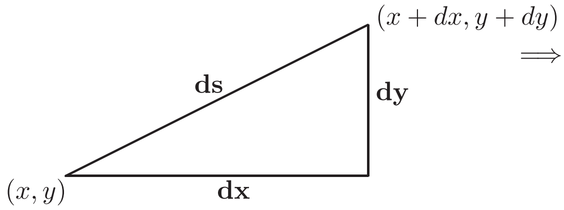
Functional Analysis describes this process

How would you actually do this integral?

3.1 What exactly is "ds"?

We normally express ds in terms of some coordinates. $s = \int_\gamma ds$ doesn't depend on having coordinates, but to work it out we would normally have a description of γ in terms of some coordinates, and we would rewrite ds in terms of these coordinates.

For example, in the ordinary 2d plane $\mathbb{R}^2$ with Cartesian coordinates



$$\Longrightarrow ds^2 = dx^2 + dy^2$$

(of course we draw small line segments, but refer to the infinitesimal limit).

In practice

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

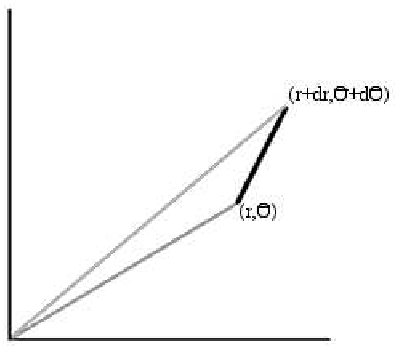
The left hand side shouldn't depend on coordinates, but the right hand side does.

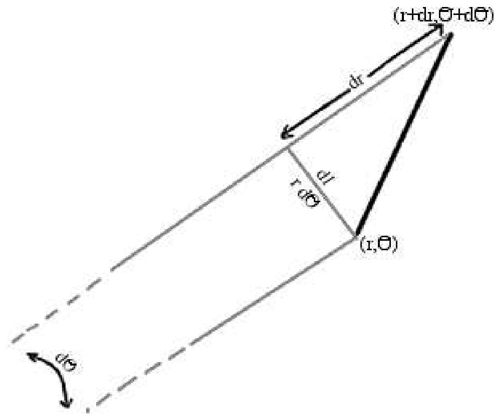
Lecture 2 ds is an infinitesimal increment along the path. In practical calculations we use coordinates. For the $\mathbb{R}^2$ example we can use Cartesian coordinates (x, y) and thus

$$ds^2 = dx^2 + dy^2$$

Another set of coordinates for $\mathbb{R}^2$ is polar coordinates (r, θ), where

$$x = r\cos\theta \qquad y = r\sin\theta$$





$$\begin{aligned} ds^2 &= dr^2 + dl^2 \\ &= dr^2 + r^2 d\theta^2 \end{aligned}$$

i.e. in these coordinates

$$ds^2 = dr^2 + r^2 d\theta^2$$

So the point is that "ds" between two infinitesimally proximate points is the same, but the expression in terms of the coordinates is different.

Of course, this isn't always the case. Here we are considering two different coordinates for the same space ($\mathbb{R}^2$), but of course we can think about other spaces.

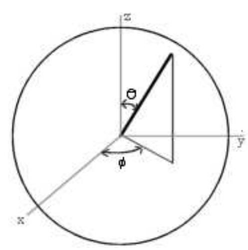
Example: S^2 - 2-sphere
Reminder: the surface of a 3d ball - 2 dimensional.
Defined by $x^2 + y^2 + z^2 = 1$ (unit sphere).

Using spherical polar coordinates:

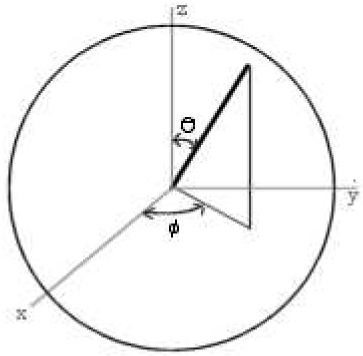
Aside: Spherical Polar Coordinates

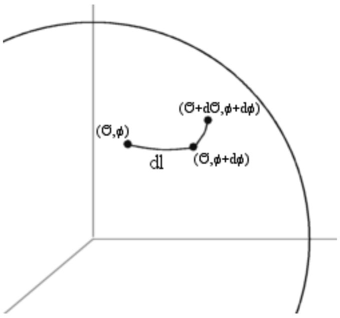
$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

where θ is the azimuthal angle, ϕ polar.



Tackling infinitesimal distances:



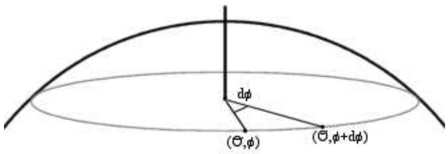


Working out dl first

Radius is $\sin\theta$ $\Rightarrow dl = \sin\theta\, d\phi$

Now taking the $d\theta$ component.

These are along a circle of longitude, all circles of longitude have radius one.



So keep θ fixed and vary ϕ

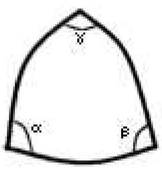
$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

This is really different from our preceding two:

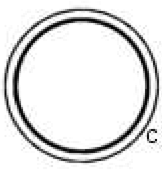
$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$\rightarrow$ these are not changes of coordinates, will change $\mathbb{R}^3$ metric to the S^2 metric.

What is different?



$\alpha + \beta + \gamma > \pi$



$2\pi r > c$

→ these are signatures of curvature.

What we want to do here is

- realise space-time is curved
- describe this curvature in a convenient way
- find an equation for the curvature of space-time

4 Coordinates & Metrics

Consider a metric space with coordinates $(x^1, x^2, \ldots, x^d)$
→ can be local coordinates
→ in GR, coords are indexed by a subscript.

We write $ds^2 = g_{ab} dx^a dx^b$ expressing the infinitesimal in terms of the coordinates.

We are using Einstein's convention, and g_{ab} is called the **metric tensor**[1], a 2 indexed object:

$$[g_{ab}]^2 = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1d} \\ g_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ g_{d1} & \cdots & \cdots & g_{dd} \end{pmatrix} \tag{3}$$

We should always take this to be symmetric:

$$g_{ab} = g_{ba}$$

Example 1:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= {dx^1}^2 + {dx^2}^2 \\ &= g_{ab}\, dx^a dx^b \end{aligned}$$

$$[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 2:

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 \qquad x^1 = r,\, x^2 = \theta \\ &= g_{ab}\, dx^a dx^b \end{aligned}$$

$$[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{pmatrix}$$

Note: It is a common abuse of notation to move between indexed and conventional symbols for coordinates, e.g.

$$[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

[1] Tensors to be defined
[2] Square brackets around a tensor mean expressing the tensor as a matrix.

Lecture 3

$$ds^2 = g_{ab}\, dx^a dx^b \tag{4}$$

This relates the length ds to infinitesimal increments in the coordinates.

Again we are using the summation convention, matching pairs of up and down indices to be summed over.

We are interested here in changes of coordinates

$$\left.\begin{aligned} [g_{ab}] &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for Cartesian} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{for Polar} \end{aligned}\right\} \text{for same } ds^2$$

Consider a change of coordinates

$$x^a \longmapsto y^b(x^a)$$

where y^b is some new set of coordinates expressible in terms of the old ones.

Example: $(r, \theta) \longmapsto (x, y)$

$$\begin{aligned} x(r,\theta) &= r\cos\theta \\ y(r,\theta) &= r\sin\theta \end{aligned}$$

There is a notation convention used here that is initially confusing but ultimately useful: we use the letters with unprimed & primed indices to mean different things, i.e. rather than write x'^a as a different coordinate to x^a, we write $x^{a'}$ (d'Inverno doesn't though).

So x^a is one set of coordinates, $x^{a'}$ is another.

Above Example implies

$$x^1 = r \qquad x^{1'} = x$$

$$x^2 = \theta \qquad x^{2'} = y$$

Coordinate change is

$$x^a \longmapsto x^{a'}(x^a)$$

Consider a change of coordinates

$$x^a \longmapsto x^{a'}$$

$$ds^2 = g_{ab}\, dx^a dx^b = g_{a'b'}\, dx^{a'} dx^{b'} \tag{5}$$

ds^2 is the same! However $[g_{a'b'}]$ is not necessarily the same as $[g_{ab}]$.

We can express $dx^{a'}$ in terms of dx^a using the Chain Rule

$$dx^{a'} = \frac{\partial x^{a'}}{\partial x^a}\, dx^a \tag{6}$$

Note: that summation converts an upper index which is below the line in a derivative, counts as a down index, i.e.

$$dx^{a'} = \sum_{a=1}^{n} \frac{\partial x^{a'}}{\partial x^a}\, dx^a$$

We aren't using it here, but a down index below a line counts as an upper index.

$$\begin{aligned}
ds^2 &= g_{ab}\, dx^a dx^b \\
&= g_{a'b'}\, dx^{a'} dx^{b'} \\
&= g_{a'b'} \left(\frac{\partial x^{a'}}{\partial x^a} dx^a\right)\left(\frac{\partial x^{b'}}{\partial x^b} dx^b\right) \\
&= \frac{\partial x^{a'}}{\partial x^a}\,\frac{\partial x^{b'}}{\partial x^b} g_{a'b'}\, dx^a dx^b \\
\Longrightarrow g_{ab} &= \frac{\partial x^{a'}}{\partial x^a}\,\frac{\partial x^{b'}}{\partial x^b}\, g_{a'b'} \qquad (7)
\end{aligned}$$

As $\frac{\partial x^a}{\partial x^b} = \delta^a_b$, by the Chain rule:

$$\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{a'}}{\partial x^b} = \delta^a_b \qquad (8)$$

$$\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^a} = \delta^{b'}_{a'} \qquad (9)$$

giving

$$g_{ab} = \frac{\partial x^{a'}}{\partial x^a}\,\frac{\partial x^{b'}}{\partial x^b}\, g_{a'b'} \qquad (10)$$

$$\begin{aligned}
\frac{\partial x^a}{\partial x^{c'}}\frac{\partial x^b}{\partial x^{d'}} g_{ab} &= \underbrace{\left(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial x^{c'}}\right)}_{\delta^{a'}_{c'}}\underbrace{\left(\frac{\partial x^{b'}}{\partial x^b}\frac{\partial x^b}{\partial x^{d'}}\right)}_{\delta^{b'}_{d'}} g_{a'b'} \\
&= g_{c'd'} \qquad (11)
\end{aligned}$$

What we have done is calculate how g_{ab} changes under a coordinate transformation (but remember ds doesn't change).

$$\begin{aligned}
x^a &\longmapsto x^{a'} \\
dx^a &\longmapsto dx^{a'} = A^{a'}_a\, dx^a \qquad \text{where } A^{a'}_a = \tfrac{\partial x^{a'}}{\partial x^a} \qquad (12) \\
g_{ab} &\longmapsto g_{a'b'} = A^a_{a'}\, A^b_{b'}\, g_{ab}
\end{aligned}$$

Example 1: $\mathbb{R}^2$ Here we change coordinates from $x^1 = x, x^2 = y$ to $x^{1'} = r, x^{2'} = \theta$.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

$$\frac{dx}{dr} = \cos\theta \qquad\qquad \frac{dx}{d\theta} = -r\sin\theta$$

$$\frac{dy}{dr} = \sin\theta \qquad\qquad \frac{dy}{d\theta} = r\cos\theta$$

$$\begin{aligned} g_{1'1'} &= \frac{\partial x^a}{\partial x^{1'}}\frac{\partial x^b}{\partial x^{1'}}\, g_{ab} \\ &= \left(\frac{\partial x^1}{\partial x^{1'}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{1'}}\right)^2 \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

$$\begin{aligned} g_{1'2'} &= \frac{\partial x^a}{\partial x^{1'}}\frac{\partial x^b}{\partial x^{2'}}\, g_{ab} \\ &= \frac{\partial x^1}{\partial x^{1'}}\,\frac{\partial x^1}{\partial x^{2'}} + \frac{\partial x^2}{\partial x^{1'}}\,\frac{\partial x^2}{\partial x^{2'}} \\ &= \frac{\partial x}{\partial r}\,\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\,\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta\,\cos\theta + r\sin\theta\,\cos\theta \\ &= 0 \end{aligned}$$

Similarily

$$g_{2'1'} = 0$$

$$\begin{aligned} g_{2'2'} &= \frac{\partial x^a}{\partial x^{2'}}\frac{\partial x^b}{\partial x^{2'}}\, g_{ab} \\ &= \left(\frac{\partial x^1}{\partial x^{2'}}\right)^2 + \left(\frac{\partial x^2}{\partial x^{2'}}\right)^2 \\ &= r^2\sin^2\theta + r^2\cos^2\theta \\ &= r^2 \end{aligned}$$

$$\Longrightarrow \boxed{[g_{a'b'}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} \qquad \text{as before.}$$

Example 2:

$$\begin{aligned} x^{1'} &= xc + ys \\ x^{2'} &= -xs + yc \end{aligned} \tag{13}$$

where

$$s^2 + c^2 = 1$$

i.e.

$$c = \cos\theta \qquad s = \sin\theta$$

If I work this out, I get:

$$g_{1'1'} = 1 \quad g_{1'2'} = 0 \quad g_{2'1'} = 0 \quad g_{2'2'} = 1$$

$\Longrightarrow$ the metric tensor remains the same - as this case is a rotation.

A transformation that leaves the metric tensor the same is called an **isometry**

$$\begin{array}{ccc} [g_{a'b'}] & = & [g_{ab}] \\ \| & & \| \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

i.e. some coordinate changes may leave the exact form of g_{ab} the same
$\longrightarrow$ these are isometries
but we are interested in all coordinate changes - general transformations.

Lecture 4 Recall

$$\begin{array}{rcccl} x^a & \longmapsto & x^{a'} & & \\ dx^a & \longmapsto & dx^{a'} & = & A^a_{a'}\, dx^a \\ g_{ab} & \longmapsto & g_{a'b'} & = & A^{a'}_a\, A^{b'}_b\, g_{ab} \end{array}$$

where

$$A^a_{a'} = \frac{\partial x^{a'}}{\partial x^a}$$

and

$$A^{a'}_a = \frac{\partial x^a}{\partial x^{a'}}$$

5 Isometries

For a given metric there may exist specific coordinate changes that leave the exact form of the metric fixed - these are **isometries**, i.e.

$$x^a \longmapsto x^{a'}$$

such that

$$[g_{ab}] = [g_{a'b'}]$$

Example:

$$\begin{array}{rcccl} x & \longmapsto & x' & = & cx + sy \\ y & \longmapsto & y' & = & -sx + cy \end{array} \tag{14}$$

where

$$c = \cos\theta, s = \sin\theta$$

$x^1 = x$, $x^2 = y$, $x^{1'} = x'$, $x^{2'} = y'$ (defining an indexed notation).

$$\frac{\partial x^{1'}}{\partial x^1} = A^{1'}_1 = c; \qquad \frac{\partial x^{2'}}{\partial x^1} = A^{2'}_1 = -s \tag{15}$$

etc. to find

$$g_{1'1'} = c^2 + s^2 = 1 \tag{16}$$

$$g_{2'2'} = 1 \tag{17}$$

$$g_{1'2'} = g_{2'1'} = 0 \tag{18}$$

$\Rightarrow$ this is an isometry on $\mathbb{R}^2$ (flat).

Example:

$$\begin{array}{rcccl} x & \longmapsto & x' & = & x + a \\ y & \longmapsto & y' & = & y + b \end{array} \tag{19}$$

a, b constant - this is also an isometry.

So isometries of $\mathbb{R}$ are translations and rotations (together they form a group called the Euclidean group - see Course 445).

Thus isometries are a special metric specific group of coordinate transformations. For now though, we want to think about *General Covariance*: the consequence of a general (smooth) coordinate change.

The idea is that fundamental equations should be expressible in a way that makes sense in all coordinate systems.

For example, in 3d flat space $\vec{F} = m\vec{a}$ is Newton's law in Cartesian coords, but $T = I\dot{w}$ is the corresponding law for rotational motion. However these two equations are expressing the same principle, they are just written in different coordinate systems. It should be possible to express them as special cases of a single equation.

We need to make the index structure explicit, i.e. we need to work with **Tensors**, indexed objects with known coordinate transform properties.

5.1 Definition of Tensors

$$
\begin{array}{rcll}
x^a & \longmapsto & x^{a'} & \\
dx^a & \longmapsto & dx^{a'} & = A^{a'}_a \, dx^a \\
g_{ab} & \longmapsto & g_{a'b'} & = A^a_{a'} \, A^b_{b'} \, g_{ab}
\end{array}
$$

More generally, define a scalar ϕ as a function of x^a such that

$$\phi(x^a) \longmapsto \phi(x^{a'})$$

The value of ϕ at a given point remains the same although the coordinate description of that point changes.

Contravariant Vector is a single indexed function of coordinates with the transformation law

$$
\begin{array}{rcll}
x^a & \longmapsto & x^{a'} & \\
v^a & \longmapsto & v^{a'} & = A^{a'}_a \, v^a
\end{array}
$$

(Contravariant vector is a name given to a vector function over space(time) with a particular transformation property.)

We have already had an example of a contravariant vector: dx^a.

Covariant Vector is a single indexed function of coordinates with the transformation law

$$
\begin{array}{rcll}
x^a & \longmapsto & x^{a'} & \\
u_a & \longmapsto & u_{a'} & = A^a_{a'} \, u_a
\end{array}
$$

The gradient of a scalar is a covariant vector:

$$\partial_a \phi = \frac{\partial \phi}{\partial x^a}$$

This is the normal gradient in $\mathbb{R}^n$

$$
\begin{aligned}
x^a &\longmapsto x^{a'} \\
\partial_a \phi &\longmapsto \partial_{a'} \phi \\
&= \frac{\partial}{\partial x^{a'}} \phi \\
&= \frac{\partial x^a}{\partial x^{a'}} \frac{\partial}{\partial x^a} \phi \\
&= A^a_{a'} \, \partial_a \phi
\end{aligned}
$$

Definition: A type (r, s) tensor is an object with r contravariant and s covariant indices, that is, it is an $r + s$ indexed function of coordinates with the transformation property

$$
\begin{array}{rcl}
x^a & \longmapsto & x^{a'} \\
T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} & \longmapsto & T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} \\
& & = A^{a'_1}_{a_1} A^{a'_2}_{a_2} \ldots A^{a'_r}_{a_r} A^{b_1}_{b'_1} A^{b_2}_{b'_2} \ldots A^{b_s}_{b'_s} \left(T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \right)
\end{array}
$$

The contravariant vector is a (1,0) tensor.
The covariant vector is a (0,1) tensor.
The metric tensor is an example of a (0,2) tensor.

Lecture 5

FAQ Is δ_{ab} a tensor? **No!**
Sometimes, particularly in applied maths, we use tensors but don't consider general transformations, i.e. strict tensors are only isometries on flat $\mathbb{R}^2$.
For flat $\mathbb{R}^3$

$$g_{ab} = \delta_{ab} \qquad [g_{ab}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\delta_{ab} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

When δ_{ab} is used often, it is really supposed to be g_{ab}, but in the context $g_{ab} = \delta_{ab}$ - restricted context.

We defined a tensor yesterday:
An (r, s) tensor has r-up indices and s-down indices and

$$\begin{aligned} T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \longmapsto \; & T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} \\ & = A^{a'_1}_{a_1} A^{a'_2}_{a_2} \ldots A^{a'_r}_{a_r} A^{b'_1}_{b_1} A^{b'_2}_{b_2} \ldots A^{b'_s}_{b_s} \left(T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \right) \end{aligned}$$

5.2 Properties of Tensors

1. **Linear** For S an (r, s) tensor, T an (r, s) tensor, α & β real, then

$$\alpha \, S^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} + \beta \, T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$$

is an (r, s) tensor.

Proof:

$$\begin{aligned} & \alpha \, S^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} + \beta \, T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \\ \longmapsto \quad & \alpha \, S^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} + \beta \, T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} \\ = \quad & \alpha \, A^{a'_1}_{a_1} A^{a'_2}_{a_2} \ldots A^{a'_r}_{a_r} A^{b_1}_{b'_1} A^{b_2}_{b'_2} \ldots A^{b_s}_{b'_s} \, S^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \\ & + \beta \, A^{a'_1}_{a_1} A^{a'_2}_{a_2} \ldots A^{a'_r}_{a_r} A^{b_1}_{b'_1} A^{b_2}_{b'_2} \ldots A^{b_s}_{b'_s} \, T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \\ = \quad & \left(A^{a'_1}_{a_1} A^{a'_2}_{a_2} \ldots A^{a'_r}_{a_r} A^{b_1}_{b'_1} A^{b_2}_{b'_2} \ldots A^{b_s}_{b'_s} \right) \left(\alpha \, S^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} + \beta \, T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \right) \end{aligned}$$

2. **Multiplication** for S an (r_1, s_1) tensor, T an (r_2, s_2) tensor, then

$$S^{a_1 a_2 \ldots a_{r_1}}{}_{b_1 b_2 \ldots b_{s_1}} \, T^{c_1 c_2 \ldots c_{r_2}}{}_{d_1 d_2 \ldots d_{s_2}}$$

is a $(r_1 + r_2, s_1 + s_2)$ tensor. Prove by checking the transformation property.

Note: Not all $(r_1 + r_2, s_1 + s_2)$ tensors are of this form. For example is U_a and V_b are covariant vectors, i.e. (0,1) tensors, then

$$T_{ab} = U_a \, V_b$$

is a (0,2) tensor, but in d-dimensions $(a = 1, \ldots, d)$ a general (0,2) tensor has d^2 components:

$$[T_{ab}] = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1d} \\ T_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ T_{d1} & \cdots & \cdots & T_{dd} \end{pmatrix}$$

$T_{ab} = U_a \, V_b$ has at most $2d$ independent components.
For $d > 2$, it is clear that a general (0,2) tensor has more degrees of freedom than a (0,2) tensor formed by multiplying two (0,1) tensors [d=2 case is slightly more subtle]

---

3. **Contraction** given a type (r, s) tensor, you can form a type $(r-1, s-1)$ tensor by summing one up index with a down index.

   Writing this down for a general tensor is notationally messy, so let's just examine examples.

   $T^{ab}{}_c$ is a (2,1) tensor, consider

$$T^{ab}{}_b = \sum_{b=1}^{d} T^{ab}{}_b$$

   Show this is a (0,1) tensor:

$$T^{ab}{}_b \longmapsto T^{a'b'}{}_{b'} = A^{a'}_a A^{b'}_b A^c_{b'} T^{ab}{}_c$$

   and using the tensor property of $T^{ab}{}_c$

$$\begin{aligned} &= A^{a'}_a \delta^c_b T^{ab}{}_c \\ &= A^{a'}_a T^{ab}{}_b \end{aligned}$$

   that is

$$V^a = T^{ab}{}_b$$

   We've also just shown that

$$V^a \longmapsto V^{a'} = A^{a'}_a V^a$$

   so we see $T^{ab}{}_b$ is a (0,1) tensor.

   Combining properties 2 & 3 together above, we can contract tensors together.

   U^a, V_b are (1,0) and (0,1) tensors respectively

$$\begin{array}{ll} U^a V_b & (1,1)\,\text{tensor} \\ U^a V_a & (0,0)\,\text{tensor} - \text{a scalar} \end{array}$$

   $g_{ab} v^b$ for example is a (1,0) tensor.

4. **Raising & Lowering** If T is an (r, s) tensor, there are lots (WRT r) of $(r-1, s+1)$ tensors by contracting with the metric. For example

$$\begin{array}{rl} U^a & (1,0)\,\text{tensor} \\ g_{ab} V^b & (0,1)\,\text{tensor} \end{array}$$

   More generally, if $T^{a_1 a_2 \dots a_r}{}_{b_1 b_2 \dots b_s}$ is an (r, s) tensor, then

$$g_{ca_i} T^{a_1 a_2 \dots a_i \dots a_r}{}_{b_1 b_2 \dots b_s}$$

   is an $(r-1, s+1)$ tensor.

   Notation is lowering: if V^a is a (1,0) tensor:

$$V_a = g_{ab} V^b$$

$$T^{a_1 a_2 \dots a_{i-1}}{}_c{}^{a_{i+1} \dots a_r}{}_{b_1 b_2 \dots b_s} = g_{ca_i} T^{a_1 a_2 \dots a_i \dots a_r}{}_{b_1 b_2 \dots b_s}$$

Lecture 6

4. **Raising & Lowering** cntd.
Define
$$T^{a_1 a_2 \dots a_{i-1}}{}_{c}{}^{a_{i+1}\dots a_r}{}_{b_1 b_2 \dots b_s} = g_{ca_i}\, T^{a_1 a_2 \dots a_i \dots a_r}{}_{b_1 b_2 \dots b_s}$$
that is, we lower the a_i index by contracting with the metric.

> **Example:**
> $$T^{a}{}_{b} = g_{bc} T^{ac}$$
> $$U_a = g_{ab} U^b$$
> Observe the following:
> $$\begin{array}{ccccccc} g_{ab}U^aU^b & = & U^aU_a & = & U_bU^b & = \text{scalar} \\ ds^2 & = & dx^a dx_a & = & g_{ab}\, dx^a dx^b & \end{array}$$

Define **raising** so that raising a lowered index is the same as "not having lowered it in the first place." This is done by defining g^{ab}, the inverse metric, by
$$g^{ab} g_{bc} = \delta^a_c$$
$$\text{that is } [g^{ab}] = [g_{ab}]^{-1}$$

> **Example:** for polar coordinated in 2D
> $$[g^{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \qquad \text{when} \qquad [g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

g^{ab} is a tensor (easy to prove).
So if U_a is a (0,1) tensor, then
$$U^a = g^{ab}\, U_b$$
is a (0,1) tensor.
More generally, given a (r, s) tensor T, we can form a $(r+1, s-1)$ tensor by contracting with g^{ab}:
$$T^{a_1 a_2 \dots a_r}{}_{b_1 b_2 \dots b_{i-1}}{}^{c}{}_{b_{i+1}\dots b_s} = g^{cb_i}\, T^{a_1 a_2 \dots a_r}{}_{b_1 b_2 \dots b_i \dots b_s}$$
If $U_a = g_{ab}\, U^b$ (†) then
$$U^a = g_{ab} U_b = g^{ab} \underbrace{(g_{bc} U^c)}_{\text{by (†)}} = \delta^a_c\, U^c = U^a$$
Good! Lowering and then raising restores the original. The notation is consistent!

5. **Symmetries** A tensor is symmetric in two indices of the same type if they can be exchanged without changing the value, that is
$$T^{a_1 \dots a_p \dots a_q \dots a_r}{}_{b_1 \dots b_s} = T^{a_1 \dots a_q \dots a_p \dots a_r}{}_{b_1 \dots b_s}$$
then T is symmetric in a_p and a_q.

> **Example:**
> g_{ab} symmetric in a and b means $g_{ab} = g_{ba} \forall a, b$
> Say $d = 3$
> $$[g_{ab}] = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$
> Then saying g_{ab} is symmetric is to say
> $$g_{12} = g_{21} \qquad g_{23} = g_{32} \qquad g_{13} = g_{31}$$

Metric tensors are always symmetric!

Note

$$g_{a'b'} = A^a{}_{a'} A^b{}_{b'} g_{ab}$$

Now

$$g_{b'a'} = A^a{}_{b'} A^b{}_{a'} g_{ab} = A^a{}_{b'} A^b{}_{a'} g_{ba}$$

by symmetry of g_{ab}
a and b are both dummy indices summed over, so it doesn't matter what we call them. So let's do a change of index, renaming a to b and vica-versa:

$$g_{b'a'} = A^b{}_{b'} A^a{}_{a'} g_{a'b'}$$

The coordinate change transformed tensor of a symmetric tensor is also symmetric!

Anti/skew symmetry is the property whereby a tensor changes sign under the exchange of two indices, for example w_{ab} is skew-symmetric if $w_{ba} = -w_{ab} \forall a, b$, so for $d = 3$:

$$[w_{ab}] = \begin{pmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{pmatrix}$$

It is easy to prove, as in the example, that symmetry and anti-symmetry are tensor properties in the sense that they are preserved by tensor transformations.

6. **Derivatives**

$$\partial_a \phi = \frac{\partial \phi}{\partial x^a} \qquad \text{is a (0,1) tensor}$$

Consider the (1,0) tensor V^a and take its derivative

$$\partial_a V^b = \frac{\partial V^b}{\partial x^a}$$

looks like it might be a (1,1) tensor. But it's not! Look what happens under a coordinate transform:

$$\begin{aligned} x^a &\longmapsto x^{a'} \\ \partial_a V^b &\longmapsto \partial_{a'} V^{b'} &&= \tfrac{\partial}{\partial x^{a'}} (V^{b'}) \\ &&&= \tfrac{\partial}{\partial x^{a'}} (A^{b'}_b V^b) \\ &&&= \tfrac{\partial x^a}{\partial x^{a'}} \tfrac{\partial}{\partial x^a} (A^{b'}_b V^b) \\ &&&= A^a{}_{a'} \tfrac{\partial}{\partial x^a} (A^{b'}_b V^b) \\ &&&= A^a{}_{a'} \left[\tfrac{\partial}{\partial x^a} (A^{b'}_b) V^b + A^{b'}_b \tfrac{\partial}{\partial x^a} (V^b) \right] \\ &&&= A^a{}_{a'} A^b{}_{b'} \tfrac{\partial}{\partial x^{a'}} (V^{b'}) + \underbrace{A^a{}_{a'} \tfrac{\partial^2 x^{b'}}{\partial x^a \partial x^b} V^b}_{\text{what's this for?}} \end{aligned}$$

This means that $\partial_a V^b$ doesn't obey the vector transformation law.

Say you had discovered the law $p = \nabla_2^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi$ in Cartesian coordinates. Expressing in tensor form:

$$p = \partial_a g^{ab} \partial_b \phi$$

Certainly, for $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $x^1 = x$, $x^2 = y$ this looks right, as it reduces to

$$\underbrace{p}_{\text{scalar}} = \partial_a \underbrace{g^{ab} \underbrace{\partial_b \phi}_{\text{(0,1) tensor}}}_{\text{(1,0) tensor}}$$

If $\partial_a g^{ab} \partial_b \phi$ was a (1,1) tensor, then $\partial_a g^{ab} \partial_b \phi$ would be a scalar, but it simply isn't! Although $p = \partial_a g^{ab} \partial_b \phi$ reduces to $p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi$ in Cartesian coordinates, it isn't true for all coordinate systems.

Note: Consider Polar coordinates in 2D

Let's work out what $\partial_a g^{ab} \partial_b \phi$ is in polar coordinates. Recall that $x^1 = r$, $x^2 = \theta$ and the metric is $[g_{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $[g^{ab}] = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$

$$\begin{aligned} \partial_a \partial^a &= \partial_1 g^{11} \partial_1 \phi + \partial_1 g^{12} \partial_2 \phi + \partial_2 g^{21} \partial_1 \phi + \partial_2 g^{22} \partial_2 \phi \\ &= \partial_r \partial_r \phi + \partial_\theta r^{-2} \partial_\theta \phi \\ &= \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \end{aligned}$$

but in fact, in polar coordinates, we already know for sure that the Laplacian in (r, θ) is

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \\ &\neq \frac{\partial}{\partial r} \left(\frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \end{aligned}$$

which we derived above.

Thus $\partial_a V^b$ is not the correct differential operator for tensors, since it itself is not a tensor, that is

$$\Delta = \nabla^2 \neq \partial_a \partial^a = \partial_a g^{ab} \partial_a$$

in general coordinates. We need a new differential operator for tensors!

Lecture 7 We have seen that the derivative of a scalar, $\partial_a \phi$ is a tensor

$$\begin{array}{rcl} x^a & \longmapsto & x^{a'} \\ \partial_a \phi & \longmapsto & \partial_{a'}\phi = A^a_{a'}\partial_a\phi \end{array}$$

(0,1) tensor in fact - covariant.

Also we saw the last day that the derivative of a contravariant vector (a (0,1) tensor) is not a tensor.

$$\partial_a v^b \neq \text{tensor}$$

Example You have seen before that the laplace operator in the 2d polar coordinates is

$$\Delta\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$\rightarrow$ not $\partial_a g^{ab}\partial_b\phi$
a naïve expectation might be that Δ is $\partial_a g^{ab}\partial_b$ in general coordinates, but it's not!

$$\partial_a \underbrace{g^{ab}\partial_b\phi}_{(1,0)\text{ tensor}} = \text{nonsense}$$

6 Covariant Derivative

We want a derivative operator acting on (1,0) tensors which gives a (1,1) tensor. This operator should
→ include differentiation
→ be linear
→ obey the Leibnitz rule (see later)
→ reduce to "just differentiating" for flat space with Cartesian coordinates.

Assume such a thing exists, i.e.

$\nabla_a V^b$ is a (1,1) tensor $\qquad\qquad$ ∇_a a linear operator

The obvious form for this

$$\nabla_b V^a = \partial_b V^a + \Gamma^a_{bc}V^c \tag{20}$$

where Γ^a_{bc} is some 3 index object whose properties are to be defined in what follows.

Γ^a_{bc} is called the **connection**.

∇_b is the **covariant derivative**. The symbol D_b is also used in texts, however we'll stick with d'Inverno's choice of ∇.

$$\nabla_b V^a \longmapsto \nabla_{b'}V^{a'} = \nabla_{b'}V^{a'} = \partial_{b'}V^{a'} + \Gamma^{a'}_{b'c'}V^{c'} \tag{21}$$

Since we require $\nabla_b V^a$ to be a tensor

$$\begin{aligned} \nabla_{b'}V^{a'} &= A^b_{b'}A^{a'}_a\nabla_b V^a && (22)\\ &= A^b_{b'}\partial_b\left(A^{a'}_a V^a\right) + \Gamma^{a'}_{b'c'}A^{c'}_c V^c && (23)\\ &= A^b_{b'}A^{a'}_a\partial_b V^a + A^b_{b'}\left(\partial_b A^{a'}_a\right)V^a + A^{c'}_c\Gamma^{a'}_{b'c'}V^c && (24)\end{aligned}$$

Put definition of ∇_b into this

$$A^b_{b'}A^{a'}_a\partial_b V^a + A^b_{b'}A^{a'}_a\Gamma^a_{bc}V^c = A^b_{b'}A^{a'}_a\partial_b V^a + A^b_{b'}\left(\partial_b A^{a'}_a\right)V^a + A^{c'}_c\Gamma^{a'}_{b'c'}V^c \tag{25}$$

canceling the first terms on each side

$$A^b_{b'}A^{a'}_a\Gamma^a_{bc}V^c = A^b_{b'}\left(\partial_b A^{a'}_a\right)V^a + A^{c'}_c\Gamma^{a'}_{b'c'}V^c \tag{26}$$

Choosing the first term on the RHS and change the summed a to summed c, and since this holds for all V^c, we can remove V^c

$$A^{c'}_{c}\,\Gamma^{a'}_{b'c'} = A^{b}_{b'}\,A^{a'}_{a}\,\Gamma^{a}_{bc} - A^{b}_{b'}\left(\partial_b A^{a'}_{c}\right) \tag{27}$$

Multiply by $A^{c}_{d'}$ and rename d' to c'

$$\Gamma^{a'}_{b'c'} = A^{b}_{b'}\,A^{c}_{c'}\,A^{a'}_{a}\,\Gamma^{a}_{bc} - A^{b}_{b'}\,A^{c}_{c'}\,\partial_b\,A^{a'}_{c} \tag{28}$$

i.e. the connection is not a tensor!

The connection is an extra structure in a space that enables the space to permit a covariant derivative - it is defined here by its transformation property - [just like a metric is a structure which allows you to define distance]. In fact, for a so-called "torsion-free metric connection," the connection is defined by the metric. This torsion-free metric connection is a very natural connection in a metric space.

Definition: The *torsion* is the antisymmetric part of the connection

$$T^{a}_{bc} = \frac{1}{2}\left(\Gamma^{a}_{bc} - \Gamma^{a}_{cb}\right) \qquad \text{is a tensor!} \tag{29}$$

Notation:

$$M_{[ab]} = \frac{1}{2}\left(M_{ab} - M_{ba}\right)$$

Square brackets around the indices imply the anti-symmetrization of a and b.
More generally:

$$M_{[ab\ldots c]} = \frac{1}{p!}\left[\sum_{\substack{\binom{a\ b\ \cdots c}{a_1 b_1 \cdots c_1} \\ \text{even} \\ \text{permutations}}} M_{a_1 b_1 \ldots c_1} \quad - \sum_{\substack{\binom{a\ b\ \cdots c}{a_1 b_1 \cdots c_1} \\ \text{odd} \\ \text{permutations}}}\right]$$

Lecture 8

Notation: Anti-symmetrization
Say $T_{ab\ldots c}$ is a tensor. The total anti/skew symmetrization is

$$T_{\underbrace{[ab\ldots c]}_{p}} = \frac{1}{p!}\left[\underbrace{\sum_{\binom{a\ b\ \cdots c}{a_1 b_1 \cdots c_1}}}_{\substack{\text{sum over}\\ \text{even permutations}}} T_{a_1 b_1 \ldots c_1} - \underbrace{\sum_{\binom{a\ b\ \cdots c}{a_1 b_1 \cdots c_1}}}_{\text{odd permutations}} T_{a_1 b_1 \ldots c_1} \right]$$

There is a theorem in algebra which points out that the decomposition of a permutation into transpositions (swap two elements) is not unique, but for a given permutation, it is either always odd or always even.

$$\binom{1234}{3142} \longmapsto \binom{1234}{3124} \longmapsto \binom{1234}{1324} \longmapsto \binom{1234}{1234}$$

3 transpositions $\Rightarrow$ odd.

Example $p = 3$

$$T_{[abc]} = \frac{1}{6}\left(T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{acb} - T_{cba}\right)$$

We can check $T_{[ab\ldots c]}$ is a tensor if $T_{ab\ldots c}$ is. Furthermore $M_{ab\ldots c} = T_{[ab\ldots c]}$ is anti/skew symmetric in any two indices (since even permutations of $ab\ldots c$ is an odd permutation of $ba\ldots c$.

Notation: Symmetrization
Defined similarly as above - $T_{(ab\ldots c)}$ is the total symmetrization of $T_{ab\ldots c}$

$$T_{(ab\ldots c)} = \frac{1}{p!} \sum_{\substack{\text{all}\\ \text{permutations}}} T_{ab\ldots c}$$

e.g. p=3

$$T_{(123)} = \frac{1}{3!}\left(T_{123} + T_{231} + T_{312} + T_{213} + T_{132} + T_{321}\right)$$

The symmetrization of a tensor is a tensor, and is symmetric under all pairwise exchanges.

Notation: Partial (Anti) Symmetrization

$$T_{[a\ldots b|c\ldots d|e\ldots f]}$$

means to antisymmetrize over the indices leaving out $c\ldots d$:

$$T_{[a|bc|d]} = \frac{1}{2}\left(T_{abcd} - T_{dbca}\right)$$

Similarly for symmetrization:

$$T_{(ab|c|d)} = \frac{1}{6}\left(T_{abcd} + T_{bdca} + T_{dacb} + T_{adcb} + T_{dbca} + T_{bacd}\right)$$

Returning to the main topic

$\nabla_a V^b$is a (1,1) tensor
$\nabla_a$is a covariant derivative

$$\nabla_a V^b = \partial_a V^b + \underbrace{\Gamma^b_{ac}}_{\substack{\text{connection}\\ \text{coefficients}}} V^c \tag{30}$$

with

$$\Gamma^{a'}_{b'c'} = A^c_{c'}\, A^b_{b'}\, A^{a'}_a\, \Gamma^a_{bc} - A^b_{b'}\, A^c_{c'}\left(\partial_b\, A^{a'}_c\right)$$

Definition: *Torsion*

$$T^c_{ab} = \Gamma^c_{[ab]}$$

We can show that this is a tensor and if a tensor is zero in one set of coordinates, it is a zero for all coordinates.

Hence we can consistently set torsion to zero

$$T^c_{ab} = 0$$

is a tensor equation accepted as a physical law by Einstein. (i.e. assumed true for simplicity - string theory says this is non-zero!)

→ We need to generalize the covariant derivative so that is acts on more general tensors.

To do this we assume covariant derivative has the Leibnitz property, that is we require

$$\nabla_a \left(U_a V^b\right) = \left(\nabla_a U_b\right) V^b + U_b \left(\nabla_a V^b\right) \tag{31}$$

$U_b V^b$ is a scalar implies

$$\nabla_a U_b V^b = \partial_b U_b V^b \tag{32}$$

$$\left(\partial_a U_b\right) V^b + U_b \partial_a V^b = \left(\nabla_a U_b\right) V^b + U_b \partial_a V^b + U_b \Gamma^b_{ac} V^c \tag{33}$$

Therefore

$$\nabla_a U_b = \partial_a U_b - \Gamma^c_{ab} U_c \tag{34}$$

This is (0,2) tensor, and everything else in (31) is a tensor, then $\nabla_a U_b$ is. You may check this explicitly.

By considering $T^{a \ldots b}{}_{c \ldots d} V_a \ldots W_b U^c \ldots Z^d$ and by apply a Leibnitz style rule recursively, we can show

$$\nabla_a T^{b_1 \ldots b_r}{}_{c_1 \ldots c_s} = \partial_a T^{b_1 \ldots b_r}{}_{c_1 \ldots c_s} + \sum_i \Gamma^{b_i}_{ap} \underbrace{T^{b_1 \ldots p \ldots b_r}{}_{c_1 \ldots c_s}}_{\text{p in ith position}} - \sum_i \Gamma^p_{ac_i} \underbrace{T^{b_1 \ldots b_r}{}_{c_1 \ldots p \ldots c_s}}_{\text{p in ith position}} \tag{35}$$

Lecture 9

$$\nabla_a T^{b_1\ldots b_r}{}_{c_1\ldots c_s} = \partial_a T^{b_1\ldots b_r}{}_{c_1\ldots c_s} + \sum_i \Gamma^{b_i}_{ap} \underbrace{T^{b_1\ldots p\ldots b_r}{}_{c_1\ldots c_s}}_{\text{p in ith position}} - \sum_i \Gamma^p_{ac_i} \underbrace{T^{b_1\ldots b_r}{}_{c_1\ldots p\ldots c_s}}_{\text{p in ith position}} \tag{36}$$

For GR, we assume torsion free

$$\Gamma^c_{[ab]} = 0 \qquad \Rightarrow \Gamma^c_{ab} = \Gamma^c_{ba}$$

7 The Metric Connection

A metric connection is a connection which is compatible with the metric.

$$\nabla_b g_{bc} = 0$$

is a restrictive property on the connection. In fact, we'll see that it defines the connection in terms of the metric and its derivatives.

This is important because without this property, out attempts to covariantize physics (i.e. rewrite in tensor form) would be plagued by order ambiguities.

Example: Laplacian in 2d

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

To write this in terms of tensors

$$\begin{aligned} \Delta\phi &= \nabla_a \partial^a \phi \qquad\qquad \underbrace{\partial^a \phi}_{\text{scalar}} = \nabla^a \phi \\ &= \nabla_a \nabla^a \phi \end{aligned}$$

Now we wonder do I mean

$$\Delta\phi = g^{ab}\nabla_a\nabla_b\phi$$

or

$$\Delta\phi = \nabla_a\nabla_b g^{ab}\phi$$

or

$$\Delta\phi = \nabla_a g^{ab}\nabla_b\phi$$

$$\nabla_a\left(g^{ab}\nabla_b\phi\right) = \left(\nabla_a g^{bc}\right)\nabla_b\phi + g^{ab}\nabla_a\nabla_b\phi$$

However $\nabla_a g^{bc} = 0$. Thus Metric Connection $\Leftarrow \nabla_a g_{bc} = 0$

$$\nabla_a \underbrace{g_{bc}g^{cd}}_{g_b^d} = (\nabla_a g_{bc})\, g^{cd} + g_{bc}\nabla_a g^{cd}$$

$$\Rightarrow \qquad \nabla_a\left(g^{ab}\nabla_b\phi\right) = g^{ab}\nabla_a\nabla_b\phi$$

Order ambiguity – you might have encountered this for covariantizing - doesn't occur for metric connections

$$\nabla_a g_{bc} \Leftrightarrow \nabla_a g^{bc} = 0$$

Again Einstein assumed a metric connection for this reason.

If we had a metric connection

$$\nabla_c\, g_{ab} = 0$$
$$\nabla_a\, g_{bc} = 0$$
$$\nabla_b\, g_{ca} = 0$$

$$
\begin{aligned}
0 &= \nabla_c\, g_{ab} &= \partial_c\, g_{ab} - \Gamma^e_{ca}\, g_{eb} - \Gamma^e_{cb}\, g_{ae} \qquad (37)\\
0 &= \nabla_a\, g_{bc} &= \partial_a\, g_{bc} - \Gamma^e_{ab}\, g_{ec} - \Gamma^e_{ac}\, g_{be} \qquad (38)\\
0 &= \nabla_b\, g_{ca} &= \partial_b\, g_{ca} - \Gamma^e_{bc}\, g_{ea} - \Gamma^e_{ba}\, g_{ce} \qquad (39)
\end{aligned}
$$

Do (38) + (39) - (37)

$$
\begin{aligned}
&\partial_a\, g_{bc} + \partial_b\, g_{ca} - \partial_c\, g_{ab}\\
&\qquad = \Gamma^e_{ab}\, g_{ec} + \Gamma^e_{ac}\, g_{be} + \Gamma^e_{bc}\, g_{ea} - \Gamma^e_{ba}\, g_{ce} - \Gamma^e_{ca}\, g_{eb} - \Gamma^e_{cb}\, g_{ae}\\
&\qquad = 2\Gamma^e_{ab}\, g_{ec}
\end{aligned}
$$

$$
\Longrightarrow \Gamma^f_{ab} = \frac{1}{2} g^{cf}\left(\partial_a\, g_{bc} + \partial_b\, g_{ca} - \partial_c\, g_{ab}\right) \qquad (40)
$$

This is an expression for the **metric connection** for a given metric, which is torsion-free as it is symmetric in a and b.

Sometimes the connection coefficient for a torsion-free metric connection is called a **Christoffel symbol of the first kind**

$$
\Gamma^f_{ab} = \left\{ \begin{matrix} f \\ ab \end{matrix} \right\}
$$
notation used in some books

And

$$
g_{af}\, \Gamma^f_{bc} = [bc, a]
$$
notation used in some books

is called the **Christoffel symbol of the second kind**.

Summary A given metric has a particularly natural connection associated with it and this is the torsion-free metric connection

$$
\Gamma^f_{ab} = \frac{1}{2} g^{cf}\left(\partial_a\, g_{bc} + \partial_b\, g_{ca} - \partial_c\, g_{ab}\right)
$$

Note

For Cartesian coordinates on flat space, $\Gamma^a_{bc} = 0$ for any a, b, c because
$g_{ab} = \begin{cases} 1 & a = b \\ 0 & \text{otherwise} \end{cases}$ and Γ^a_{ab} involves derivatives.

Lecture 10

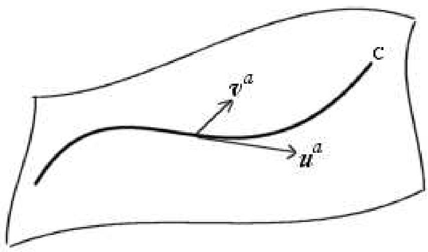
Definition: The torsion-free metric derivative is defined as

$$\begin{aligned}\Gamma^c_{[ab]} &= 0 && \leftrightarrow \text{torsion-free} \\ \nabla_a g_{bc} &= 0 && \leftrightarrow \text{metric is a covariant constant}\end{aligned}$$

$$\Gamma^f_{ab} = \frac{1}{2} g^{cf} \left(\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab}\right)$$

8 Parallel Transport

This is covariantization of the notion of constant along a curve.



Aside "Locally at least"
This means that while there may not be a coordinate system that works over the whole space (for example a sphere has coordinate singularities). We can always find a good coordinate system in a neighbourhood of a given point.
For example at $\theta = 0, \phi$ makes no sense on a 2-sphere, but we are ignoring global issues here.
The subject of differential geometry is about stitching together facts based on local coordinates into global statements. In GR, it is usually enough to work locally.

Locally at least, there are coordinates x^a and we can parametrize the curve in terms of these coordinates; $c = x^a(t)$, where t is some parameter often referred to as an affine parameter.

Choose

$$\begin{aligned} p &= \vec{x}(t=0) \\ &= (x^1(0), x^2(0), \ldots, x^d(0)) \end{aligned}$$

where d is the dimension of the space. A tangent to the curve is [3]

$$U^a = \frac{dx^a}{dt}$$

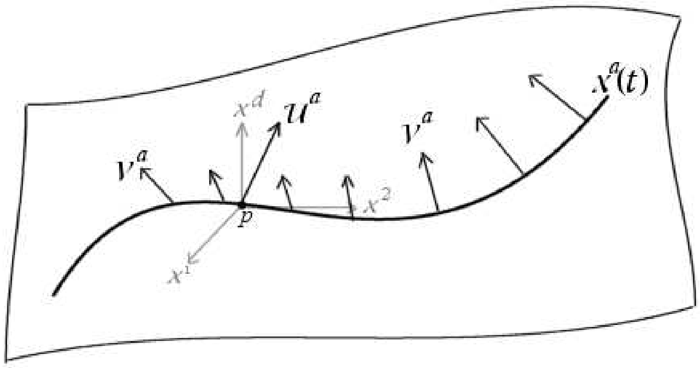
A vector $V^a(t)$ defined along the curve is said to be **parallel transported** if it satisfies

$$U^a \nabla_a V^b = 0$$

Conversely, we use the definition actively: if we are given a vector V^a at p, the vector $V^a(t)$ is the parallel transport of V^a if $V^a(0) = V^a$ and $U^a \nabla_a V^b = 0$

So parallel transport defines a vector V^a everywhere along the curve. Of course, V^a can be regarded as a function of coordinates $V^a(x^b)|_{x^b \text{ on the curve}}$ or as a function of t: $V^a(x^b(t))$

[3] the following is actually the definition of derivatives



V^a evaluated on the curve by solving differential equations.

It is similar to requiring V^a to be constant on the curve, that is

$$\frac{d}{dt} V^b = 0$$

Using the Chain rule:

$$\frac{\partial x^a}{\partial t} \frac{\partial}{\partial x^a} V^b = 0 \qquad\qquad U^a\, \partial_a\, V^b = 0$$

$U^a\, \partial_a\, V^b = 0$ is NOT covariant, but $U^a\, \nabla_a\, V^b = 0$ is!

9 Geodesic Equation

This is used to generalize the notion of a straight line.

Given parallel transport, note that a natural vector on a curve is the tangent vector. Generically, of course, the tangent vector isn't parallel transported.

A geodesic is a curve $x^a(t)$ such that

$$U^a \nabla_a U^b = 0$$
$$U^a = \frac{dx^a}{dt}$$

Note that here is an equation for a curve. Writing this out, the geodesic equation is

$$\frac{dx^a}{dt} \frac{\partial}{\partial x^a} \frac{dx^b}{dt} + \frac{dx^a}{dt} \frac{dx^c}{dt} \Gamma^b_{ac} = 0$$
$$\frac{d^2 x^b}{dt^2} + \Gamma^b_{ac} \frac{dx^a}{dt} \frac{dx^c}{dt} = 0 \qquad (41)$$

(this is the Monge-Ampère equation).

Start with a point and tangent to it, and get a curve! Note that the norm[4] of U^a is preserved:

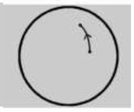
$$U^b \nabla_b (U^a U_a) = \underbrace{\left(U^b \nabla_b U^a\right)}_{\text{0 by geodesic}} U_a + U^a U^b \nabla_n U_a$$
$$= U^a \underbrace{\left(U^b \nabla_b U^c\right)}_{\text{geodesic}} g_{ac}$$
$$= 0$$

[4]the norm is the scalar product of U with itself, i.e. $N = U^a U_a$

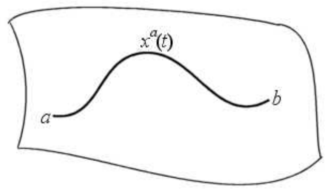
The correct generalization of a straight line is the curve of shortest distance, and in fact a geodesic between two points *is* the shortest curve between them.

Examples:

1. A straight line in flat space
2. A great circle on a sphere



$$ds^2 = g_{ab}dx^a dx^b$$
$$\Rightarrow \quad I = \int_a^b ds$$
$$= \int_a^b \sqrt{g_{ab}dx^a dx^b}$$



While is can be done, this calculation is tricky. It's easier to start by convincing yourself that a curve minimizing the distance also minimizes the integrated square distance, so we replace

$$I = \int_{t_1}^{t_2} \sqrt{g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}}dt \tag{42}$$

by

$$I' = \int_{t_1}^{t_2} g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}dt \tag{43}$$

Claim: A curve minimizing I' minimizes I.

Lecture 11 A Geodesic is a curve which parallel transports its tangents.

$$U^a = \frac{dx^a}{dt}$$
$$U^a \nabla_a U^b = 0$$
$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} = 0$$

A geodesic is a shortest path. We shall demonstrate this by showing it extremizes

$$I = \int_a^b \underbrace{\sqrt{g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}}}_{\sqrt{\frac{ds^2}{dt^2}}}\,dt$$

This relies on the principle that a path extremizing the integrated square distance extremizes the balance. Note that the extremum should be either a minimum or a saddle.

$$I' = \int_a^b \underbrace{g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}}_{L}\,dt$$

This looks like the action $S = \int L\,dt$, so we may extremize by solving the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0 \tag{44}$$
$$\Rightarrow \qquad \frac{d}{dt}\left(2g_{bc}\frac{dx^b}{dt}\right) - \frac{\partial}{\partial x^c}\left(g_{ab}\dot{x}^a\dot{x}^b\right) = 0 \tag{45}$$

Notation Use the common notation for derivatives:

$$g_{ab,c} := \frac{\partial g_{ab}}{\partial x^c} = \partial_c\, g_{ab}$$

$$\Rightarrow \qquad 2g_{bc}\frac{d^2x^b}{dt^2} + 2\frac{dg_{ac}}{dt}\dot{x}^b - g_{ab,c}\dot{x}^a\dot{x}^b = 0$$

Taking the last term from above

$$\frac{d}{dt}g_{bc} = \frac{dx^d}{dt}\frac{d}{dx^d}g_{bc}$$

by the chain rule, so

$$\Rightarrow \qquad 2g_{bc}\frac{d^2x^b}{dt^2} + \left(2g_{bc,a} - g_{ab,c}\right)\dot{x}^a\dot{x}^b = 0 \tag{46}$$

We symmetrize in a and b, that is rename $a \leftrightarrow b$

$$g_{bc,a}\dot{x}^a\dot{x}^b = g_{ac,b}\dot{x}^b\dot{x}^a = g_{ac,b}\dot{x}^a\dot{x}^b$$

The equation becomes

$$\Rightarrow \qquad 2g_{bc}\ddot{x}^b + \left(g_{bc,a} + g_{ac,b} - g_{ab,c}\right)\dot{x}^b\dot{x}^a = 0 \tag{47}$$

Now simply multiply across by $\frac{1}{2}g^{ce}$, and noting the fact that $\frac{1}{2}g^{ce}2g_{cb} = \delta^e_b$

$$\Rightarrow \qquad \boxed{\frac{1}{2}g^{ec}\left(g_{bc,a} + g_{ac,b} - g_{ab,c}\right) := \Gamma^e_{ab}} \tag{48}$$

Thus equation (47) may be expressed as:

$$\boxed{\ddot{x}^e + \Gamma^e_{ab}\dot{x}^a\dot{x}^b = 0} \tag{49}$$

This is like $a = 0$, for a Newton particle with no force; $a = \ddot{x}^e = 0$ is not a covariant equation.

10 The Curvature or Riemann Tensor[4]

$$\nabla_a \nabla_b U_c - \nabla_b \nabla_a U_c = R_{abcd} U^d \tag{50}$$

Unlike ordinary derivatives, the covariant derivative doesn't commute, since the partial term ∂_a from ∇_a acts on the connection on ∇_b. Relabelling quantifies this failure[5].

We can find an explicit formula for R_{abcd} from the definition

$$\overbrace{\nabla_a}^{(\dagger)} \underbrace{\left(\partial_b U_c - \Gamma^e_{bc} U_e\right)}_{(\ddagger)} - \nabla_b \left(\partial_b U_c - \Gamma^e_{ac} U_e\right) = R_{abcd} U^d$$

Remember the stuff in round brackets is a (0,2) tensor, so (†) has two connection terms acting on (‡).

$$\begin{aligned} R_{abcd}U^d &= \partial_a\left(\partial_b U_c - \Gamma^e_{bc}U_e\right) - \Gamma^d_{ab}\left(\partial_d U_c - \Gamma^e_{dc}U_e\right) - \Gamma^d_{ac}\left(\partial_b U_d - \Gamma^e_{bd}U_e\right) \\ &\quad - \partial_b\left(\partial_a U_c - \Gamma^e_{ac}U_e\right) + \Gamma^d_{ba}\left(\partial_d U_c - \Gamma^e_{dc}U_e\right) + \Gamma^d_{bc}\left(\partial_a U_d - \Gamma^e_{ad}U_e\right) \qquad (51) \\ &= U_{c,ba} - \Gamma^e_{bc,a}U_e - \Gamma^e_{bc}U_{e,a} - \Gamma^d_{ab}U_{c,d} + \Gamma^d_{ab}\Gamma^e_{dc}U_e - \Gamma^d_{ac}U_{d,b} + \Gamma^d_{ac}\Gamma^e_{bd}U_e \\ &\quad - U_{c,ba} + \Gamma^e_{ac,b}U_e + \Gamma^e_{ac}U_{e,b} + \Gamma^d_{ba}U_{c,d} - \Gamma^d_{ab}\Gamma^e_{dc}U_e + \Gamma^d_{bc}U_{d,a} - \Gamma^d_{bc}\Gamma^e_{ad}U_e \qquad (52) \\ &= -\Gamma^e_{bc,a}U_e + \Gamma^d_{ab}\Gamma^e_{dc}U_e + \Gamma^e_{ac,b}U_e - \Gamma^d_{ab}\Gamma^e_{dc}U_e - \Gamma^d_{bc}\Gamma^e_{ad}U_e + \Gamma^d_{ac}\Gamma^e_{bd}U_e \qquad (53) \end{aligned}$$

so

$$\begin{aligned} R_{abce}U^e &= R_{abc}{}^e U_e \qquad (54) \\ &= \left[\Gamma^e_{ac,b} - \Gamma^e_{bc,a} + \Gamma^d_{ac}\Gamma^e_{bd} - \Gamma^d_{bc}\Gamma^e_{ad}\right] U_e \qquad (55) \end{aligned}$$

This is true for all U_e, so

$$\Rightarrow \quad \boxed{R_{abc}{}^e = \partial_b \Gamma^e_{bc} - \partial_a \Gamma^e_{bc} + \Gamma^d_{ac}\Gamma^e_{bd} - \Gamma^d_{bc}\Gamma^e_{ad}} \tag{56}$$

Therefore the Riemann tensor depends on the coordinates and its derivatives, or equivalently on the metric and its first and second derivatives.

[4]The following obeys the standards (signs) set in Misner-Wheeler-Thorne (1972)
[5]like F_{ab} in field theory