Solutions to 123 Christmas exam

1. Answer is D. $(x, y) = (2, 9)$ because $3(6x - 2y) + 2(-5x + 3y) = -18 + 34$ so $8x = 16$ and $x = 2$, substitute in and get $y = 9$.

2. The answer is B. $x^2 - 5x - 14 = (x - 7)(x + 2)$ and so write

$$\frac{1}{x^2 - 5x - 14} = \frac{A}{x - 7} + \frac{B}{x + 2}$$

multiply across by $x^2 - 5x - 14$

$$1 = A(x + 2) + B(x - 7)$$

and then substitute $x = 7$ for $1 = 9A$ and $x = -2$ for $1 = -9B$.

3. The answer is B. The binomial expansion is

$$(2 - 4x)^8 = 2^8 + \binom{8}{1} 2^7(-4x) + \binom{8}{2} 2^6(-4x)^2 + \binom{8}{3} 2^5(-4x)^3 + \dots$$

and so the x^3 term is

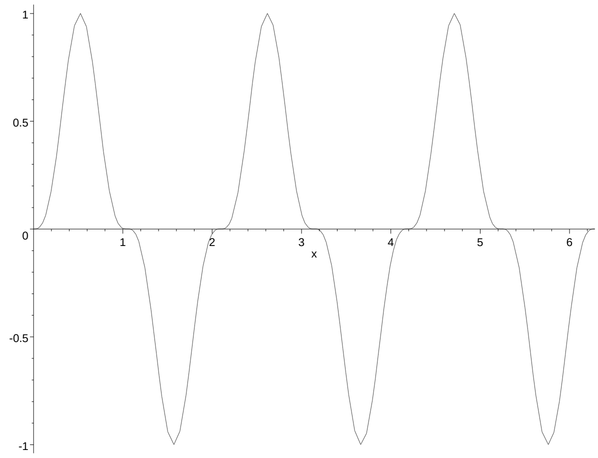
$$\binom{8}{3} 2^5(-4x)^3 = -\frac{8 \times 7 \times 6}{1 \times 2 \times 3} \times 2^5 \times 4^3 \times x^3 = -114688x^3$$

4. The answer is E. The total number of different choices of six objects out of 36 is

$$\binom{36}{6} = 1947792$$

and so at 50p each, that makes £$973, 896$.

5. The answer is C. The point is that $\sin^3 x$ has the same period as $\sin x$ because $(-\sin x)^3 = -(\sin x)^3$ and $\sin x$ has period 2π so $\sin^3 3x$ has period $2\pi/3$. The graph of $\sin^3 3x$ is given below.



6. The answer is A. The formulae to use are

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \sin B \cos A \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

and so $\sin 3x = \sin(2x + x) = \sin 2x \cos x + \sin x \cos 2x$ and then, again by the formulae above, $\cos 2x = \cos^2 x - \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$. Substituting these in gives the answer.

7. The answer is D. You use the formula

$$\log_a b = \frac{\log_c b}{\log_c a}$$

so, either,

$$\log_5 6 = \frac{\log_e 6}{\log_e 5} = \frac{1.791759469}{1.609437912} \approx 1.11328$$

or

$$\log_5 6 = \frac{\log_{10} 6}{\log_{10} 5} = \frac{.7781512503}{.6989700041} \approx 1.11328$$

8. The answer is B. There are two obvious routes to the answer, first, from the doubling time, t_d, you can work out the rate:

$$r = \frac{\log_e 2}{t_d} \Rightarrow r = \frac{.693147}{5} = .138629$$

and then use the population growth equation

$$P(t) = P_0 e^{rt}$$

to give

$$P(24) = 1000e^{24(.138629)} \approx 27858$$

or reasoning more directly, you could say, well, the population increases by a factor of two in five hours, so it increases by a factor of $2^{24/5}$ in 24 hours and $2^{24/5} = 27.858$.

9. The answer is E. Lets look at each piece on its own:

$$\frac{d}{dx}\frac{x^2}{x+1} = \frac{(x+1)2x - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}$$

by the quotient rule.

$$\frac{d}{dx}xe^{x-1} = e^{x-1} + xe^{x-1}$$

by the product rule and finally, $\log x^2 = 2\log x$ gives

$$\frac{d}{dx}\log x^2 = \frac{2}{x}$$

so

$$f'(x) = \frac{x^2+2x}{(x+1)^2} + e^{x-1} + xe^{x-1} - \frac{2}{x}$$

and substituting in $x = 1$ gives the answer.

10. The answer is D. If $y = \log u$ where $u = \tan x$ then

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\tan x}$$

and

$$\frac{du}{dx} = \sec^2 x = \frac{1}{\cos^2 x}$$

and so the answer follows by the chain rule.

11 The answer is B.

$$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-2)(x-1)$$

and so the graph has extrema at $x = 1$ and $x = 2$. Either by noting that the graph is positive for large x or by finding the second derivative

$$\frac{d^2y}{dx^2} = 12x - 18$$

so $f''(1) < 0$ but $f''(2) > 0$ where $y = f(x)$, it follows that $x = 2$ is the minimum and $x = 1$ is the maximum.

12. The answer is C. because $2 + 3x - 2x^2 = 0$ for $x = -1/2$ and $x = 2$ and y is clearly negative for large x, or, $y = 1/2$ for $x = 0$.