Solns to 123 Problem Sheet Hillary 2[1]
24 January 2001

1. $f(x) = \log(1 + x^2 + 3x^3)$. Using the chain rule work out $f'(x)$. Using the quotient rule work out $f''(1)$

Answer:

$$\frac{d}{dx}(1 + x^2 + 3x^3) = 2x + 9x^2$$

and

$$\frac{d}{du}\log u = \frac{1}{u}$$

therefore

$$f'(x) = \frac{2x + 9x^2}{1 + x^2 + 3x^3}.$$

Next

$$\frac{d}{dx}(2x + 9x^2) = 2 + 18x$$

so the quotient rule gives

$$f''(x) = \frac{(1 + x^2 + 3x^3)(2 + 18x) - (2x + 9x^2)^2}{(1 + x^2 + 3x^3)^2}$$

and hence

$$f''(1) = \frac{(1 + 1 + 3)(2 + 18) - (2 + 9)^2}{(1 + 1 + 3)^2} = -\frac{21}{25}$$

2. Differenciate $\ln\tan x$ with respect to x.

Answer: Once again use the chain rule and the fact that

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$$

to give

$$\frac{d}{dx}\ln\tan x = \frac{1}{\tan x \cos^2 x} = \frac{1}{\sin x \cos x}$$

I asked this before in the Christmas exam.

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3. Use the chain rule and product rule to differenciate $y = xe^{-x^2}$ and find the extrema.

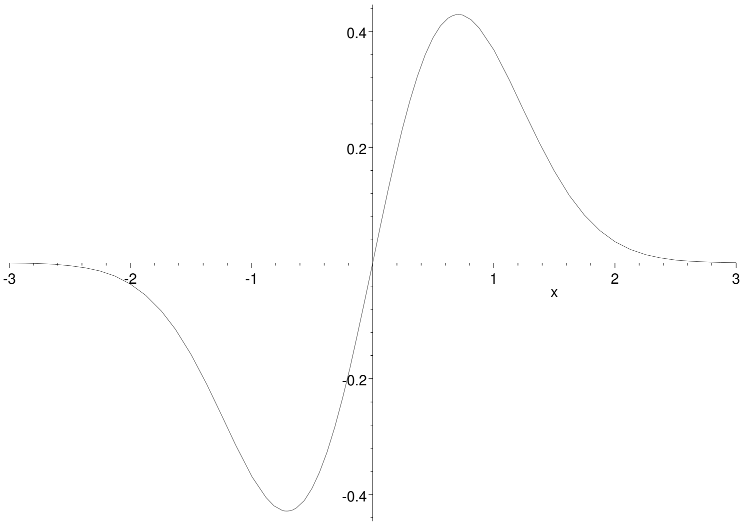
Answer:

$$\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$$

and so

$$\frac{d}{dx}xe^{-x^2} = -2x^2e^{-x^2} + e^{-x^2}.$$

This means the differencial has zeros when $x = \pm 1/\sqrt{2}$ and these are the extrema. I have discussed this example before as well. The graph is below.



4. Find the extrema of $y = 2x^3 + 3x^2 - 12x + 9$ and say weather they are maxima or minima.

Answer: So

$$\frac{dy}{dx} = 6x^2 + 6x - 12 = 6(x-1)(x+2)$$

and hence the extrema are at $x = 1$ and $x = -2$.

$$\frac{d^2y}{dx^2} = 12x + 6$$

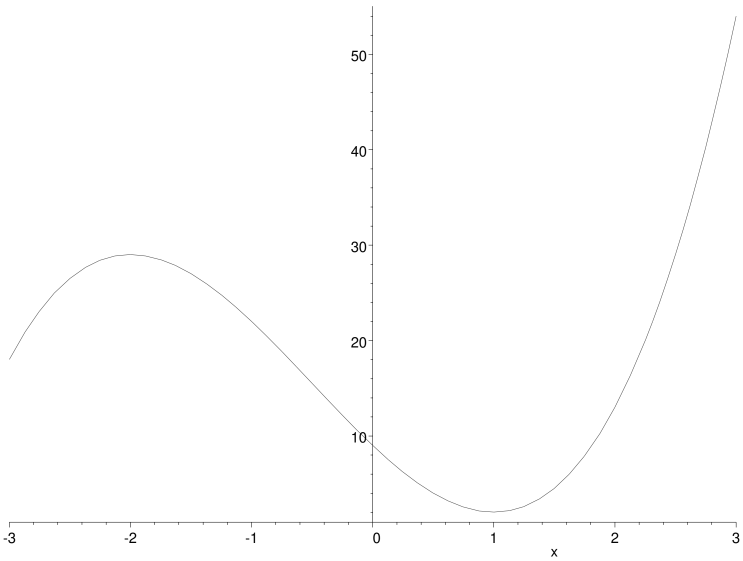
so

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 18 > 0$$

and that point is a minimum.

$$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -18 < 0$$

and that point is a maximum. The graph is below.



5. Using l'Hôpidal's rule work out

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{x}$$

Answer: Well the numerator and denominator are both zero for $x=0$. This means we apply l'Hopidal's rule:

$$\frac{d}{dx}(e^{2x}-e^{-2x}) = 2(e^{2x}+e^{-2x})$$

and so

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{x} = 2\lim_{x\to 0}(e^{2x}+e^{-2x}) = 4$$

6. Calculate the Taylor expansion of $\cot x$ around $x=\pi/2$ up to $O(h^3)$.

Answer: Again this is just an epic[2] in differenciation:

$$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$$

[2]I won't ask any Taylor series question this long in the exam

and

$$\frac{d^2}{dx^2}\cot x = -\frac{d}{dx}\frac{1}{\sin^2 x} = \frac{2\cos x}{\sin^3 x}$$

and finally

$$\frac{d^3}{dx^3}\cot x = \frac{d}{dx}\frac{2\cos x}{\sin^3 x} = -2\frac{\sin^4 x + 3\cos^2 x \sin^2}{\sin^4 x}.$$

Now $\sin\pi/2 = 1$ and $\cos\pi/2 = 0$ means that

$$\cot\left(\pi/2 + h\right) = -h - \frac{1}{3}h^3 + O(h^4)$$