123 Problem Sheet Hillary 1[1]

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1. Use L'Hôpidal's rule to calculate

$$\lim_{x\to 1} \frac{x\log_e x}{1-x} \tag{1}$$

$$\lim_{x\to \pi/2} \frac{\sin x - 1}{(x-\pi/2)^2} \tag{2}$$

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{\sqrt{1-x}} \tag{3}$$

$$\lim_{x\to 0} \frac{\log_e \cos x}{\sin^2 x} \tag{4}$$

$$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}. \tag{5}$$

If these limits are taking you a long time, then leave them after doing one or two and go on to the Taylor series questions.

(1) $\log_e 1 = 0$ and $1-x$ is also zero when $x=1$ so differenciate top and bottom. Using the product rule

$$\frac{d}{dx}x\log_e x = \log_e x + x\frac{1}{x} = \log_e x + 1$$

and

$$\frac{d}{dx}(1-x) = -1$$

so

$$\lim_{x\to 1} \frac{x\log_e x}{1-x} = \lim_{x\to 1}(-\log_e x - 1) = -1$$

(2) $\sin \pi/2 = 1$ so again, this limit looks like zero over zero and so we differenciate.

$$\frac{d}{dx}(\sin x - 1) = \cos x$$

and

$$\frac{d}{dx}(x-\pi/2)^2 = 2(x-\pi/2)$$

Thus

$$\lim_{x\to \pi/2} \frac{\sin x - 1}{(x-\pi/2)^2} = \lim_{x\to \pi/2} \frac{\cos x}{2(x-\pi/2)}$$

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which is still zero over zero and so we differenciate again. The differential of the $\cos x$ gives $-\sin x$ and the differential of the bottom gives 2, this

$$\lim_{x\to\pi/2}\frac{\sin x-1}{(x-\pi/2)^2}=\lim_{x\to\pi/2}\frac{\cos x}{2(x-\pi/2)}=\lim_{x\to\pi/2}\frac{-\sin x}{2}=-\frac{1}{2}$$

(3) This time

$$\frac{d}{dx}(1-\sqrt{x})=-\frac{1}{2\sqrt{x}}$$

and

$$\frac{d}{dx}\sqrt{1-x}=-\frac{1}{2\sqrt{1-x}}$$

so

$$\lim_{x\to1}\frac{1-\sqrt{x}}{\sqrt{1-x}}=\lim_{x\to1}=-\frac{\sqrt{1-x}}{\sqrt{x}}=0.$$

(4)

$$\frac{d}{dx}\log_e\cos x=-\frac{\sin x}{\cos x}$$

and

$$\frac{d}{dx}\sin^2 x=2\sin x\cos x$$

so

$$\lim_{x\to0}\frac{\log_e\cos x}{\sin^2 x}=\lim_{x\to0}\frac{-1}{2\cos^2 x}=-\frac{1}{2}$$

(5)

$$\lim_{x\to1}\frac{\sqrt{x+3}-2}{1-x}=\lim_{x\to1}\frac{-1}{2\sqrt{x+3}}=-\frac{1}{4}$$

2. Find the first three terms of the Taylor expansion of $\tan x$ around $x=0$.

So to do this, lets first work out the relevant differentials:

$$\frac{d}{dx}\tan x=\frac{1}{\cos^2 x}$$

and if $x=0$ this is one. Furthermore

$$\frac{d^2}{dx^2}\tan x=\frac{d}{dx}\frac{1}{\cos^2 x}=-2\frac{\sin x}{\cos^3 x}$$

and if $x=0$ this is zero. Thus the first three terms are $\tan x=-x+O(x^3)$. because the constant part and the x^2 term are zero. Often when people say the first three terms they mean the first three non-zero terms, that would be a very long question in this case and would give

$$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+O(x^6)$$

3. Find the first three term of the Taylor expansion of $1/x$ around $x = 1$.

So, let $x = 1 + h$. If $x = 1$ $1/x = 1$.

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

and this means

$$\left.\frac{d}{dx}\frac{1}{x}\right|_{x=1} = -1$$

As to the secod derivative

$$\frac{d^2}{dx^2}\frac{1}{x} = 2\frac{1}{x^3}$$

and this means

$$\left.\frac{d^2}{dx^2}\frac{1}{x}\right|_{x=1} = 2$$

and the Taylor series near $x = 1$ is

$$\frac{1}{x} = 1 + h + h^2 + O(h^3)$$

4. Expand

$$\frac{1}{\sqrt{1+h}}$$

for small h. Include terms of order two in h but stop there.

$$\frac{1}{\sqrt{1+h}} = 1 - \frac{1}{2}h + \frac{3}{8}h^2 + O(h^3)$$

Again, differenciate and evaluate.

5. Expand $(2 + x)^{1/3}$ near $x = 2$. What is the third order term.

Let $x = 2 + h$. The third order term in the expand of $f(x)$ for $x = a + h$ is the term with h^3 in it. It is third order in h, that is, h is to the third power. The third order term is

$$\frac{1}{3!}\left.\frac{d^3}{dx^3}f(x)\right|_{x=a} h^3$$

Now

$$\frac{d}{dx}(2+x)^{1/3} = \frac{1}{3}(2+x)^{-2/3}$$

so

$$\frac{d^2}{dx^2}(2+x)^{1/3} = \frac{1}{3}\left(\frac{-2}{3}\right)(2+x)^{-5/3}$$

and finally

$$\frac{d^3}{dx^3}(2+x)^{1/3} = \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-8}{3}\right)(2+x)^{-5/3}$$

and so

$$\left.\frac{d^3}{dx^3}(2+x)^{1/3}\right|_{x=2} = \frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)(2+2)^{-5/3} = \frac{10}{27}4^{-8/3}$$

and this means that the answer is $(5/81)4^{-8/3}h^3$. A neater way of writting that would be to write $4^{-8/3} = 4^{-3}4^{1/3} = (4^{1/3})/64$ giving

$$\frac{5}{5184}4^{1/3}h^3.$$