123. Summer 2001. Solutions

Long Questions

1. Rewrite

$$\frac{1}{(x-1)(x+3)(x+2)}$$

in the form

$$\frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{x+2}$$

Answer So write down the equation

$$\frac{1}{(x-1)(x+3)(x+2)}=\frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{x+2} \tag{1}$$

and then multiply across by $(x-1)(x+3)(x+2)$. This gives

$$1=A(x+3)(x+2)+B(x-1)(x+2)+C(x-1)(x+3) \tag{2}$$

choosing $x=1$ gives $1=12A$, $x=-3$ gives $1=4B$ and $x=-2$ gives $1=-3C$ and so

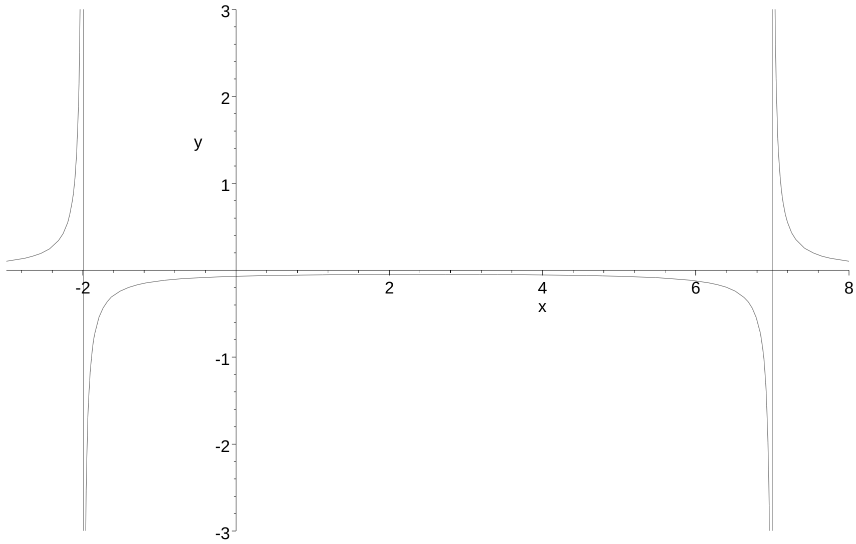
$$\frac{1}{(x-1)(x+3)(x+2)}=\frac{1}{12(x-1)}+\frac{1}{4(x+3)}-\frac{1}{3(x+2)} \tag{3}$$

2. Sketch the graph of

$$y=\frac{2}{x^2-5x-14}$$

showing clearly any asymptotes.

Answer First solve $x^2-5x-14=0$. This has solutions $x=-2$ and $x=7$. The asymptotes are located at these points. The graph is positive for $x<-2$ and $x>7$ and negative for $-2<x<7$. The graph is



3. Write down the first three non-zero terms of the Taylor expansion of $x\cos x$.

Answer This question could have been better phrased and was marked accordingly. The first three non-zero terms of the Taylor expansion of $x \cos x$ *about zero* are given by the formula

$$f(x) = f(x) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots + \frac{1}{n!}f^{(n)}(0)x^n + \ldots, \tag{4}$$

so, $f(x) = x\cos(x)$. $f(0) = 0$, $f'(x) = \cos x - x\sin x$, thus, $f'(0) = 1$. $f''(x) = -2\sin x - x\cos x$ and $f''(0) = 0$. $f'''(x) = -3\cos x + x\sin x$ and so $f'''(0) = -3$. $f^{(4)}(x) = 4\sin x + x\cos x$ and so $f^{(4)}(0) = 0$. Finally, $f^{(5)}(x) = 5\cos x - x\sin x$ so $f^{(5)}(0) = 5$. This means

$$f(x) = x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + \ldots \tag{5}$$

is the first three non-zero terms.

Short Questions

1. What are the solutions to $6x^2 + x - 2 = 0$?

A. $x = \frac{1}{2}$, $x = -\frac{2}{3}$ B. $x = -\frac{1}{2}$, $x = -\frac{2}{3}$ C. $x = \frac{1}{3}$, $x = -1$

D. $x = -\frac{1}{3}$, $x = 1$ E. $x = -2$, $x = 3$

Answer is A. Either factorize

$$0 = 6x^2 + x - 2 = (3x + 2)(2x - 1) \tag{6}$$

or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{49}}{2} \tag{7}$$

2. In the New York Lotto there are 51 balls and six are drawn. A set of six numbers costs 50 cents, but the minimum you can play is two sets of numbers for a dollar. If you have two (different) sets of numbers you chance of winning the match six prize is one in

A. 5 B. 6483405600 C. 9004730 D. 391510 E. 15002320

Answer is C. So the number of combinations is 51 choose 6.

$$\binom{51}{6} = \frac{51 \times 50 \times 49 \times 48 \times 47 \times 46}{6 \times 5 \times 4 \times 3 \times 2} = 18009460 \tag{8}$$

and halve it to get 9004730.

3. 72° in radians is

A. 2π B. π C. $2\pi/3$ D. $\pi/2$ E. $2\pi/5$

Answer is E. So $360/72 = 5$ so this is a fifth of a circle. The full circle is 2π and so a fifth of a circle is $2\pi/5$.

4. What is the period of $2\sin 3x \cos 3x$

A. π B. $\pi/2$ C. $\pi/3$ D. $\pi/4$ E. $\pi/5$

Answer is C. First $2\sin 3x \cos 3x = \sin 6x$. $\sin x$ has period 2π so $\sin 6x$ has period $\pi/3$.

5 What is

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 6}{9x^2 - 2x + 5}$$

A. $-\infty$ B. ∞ C. 1 D. $\frac{1}{3}$ E. $\frac{6}{5}$

Answer is D. Divide by the highest power of x:

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 6}{9x^2 - 2x + 5} = \lim_{x\to\infty} \frac{3 - 2/x + 6/x^2}{9 - 2/x + 5/x^2} = \frac{3}{9} = \frac{1}{3} \tag{9}$$

6. A population halves every five hours, if you begin with a population of 1000, after 24 hours the population will be approximately

A. 1155 B. 27858 C. 36 D. 12 E. 866

Answer is C. There are two obvious routes to the answer, first, from the halving time, t_h, you can work out the rate:

$$r = -\frac{\log_e 2}{t_h} \Rightarrow r = -\frac{.693147}{5} = -.138629$$

and then use the population growth equation

$$P(t) = P_0 e^{rt}$$

to give

$$P(24) = 1000e^{24(-.138629)} \approx 36$$

or reasoning more directly, you could say, well, the population decreases by a factor of a half in five hours, so it decreases by a factor of $(1/2)^{24/5}$ in 24 hours and $(1/2)^{24/5} = .03589$.

7. What is the mimimum value of $x^2 - 4x + 4$

A. -2 B. -1 C. 0 D. 1 E. 2

Answer is C. So the minimum occurs when

$$\frac{d}{dx}(x^2 - 4x + 4) = 0 \tag{10}$$

and

$$\frac{d}{dx}(x^2 - 4x + 4) = 2x - 4 \tag{11}$$

so the minumum occurs at $x = 2$. Now, if $x = 2$ then $x^2 - 4x + 4 = 0$

8. For $f(x) = \log_e(1 + x^3)^3$, what is $f'(1)$

A. $\frac{9}{2}$ B. -3 C. $3\log_e 2$ D. $9\log_e 2$ E. 9

Answer is A. First do the differenciation. You need to use the chain rule. Using the chain rule

$$\frac{d}{dx}(1 + x^3)^3 = 9x^2(1 + x^3)^2 \tag{12}$$

and then using the chain rule on the whole lot

$$f'(x) = \frac{d}{dx}\log_e(1 + x^3)^3 = \frac{9x^2}{1 + x^3} \tag{13}$$

so

$$f'(1) = \frac{9}{2} \tag{14}$$

9 What is

$$\lim_{x \to 0} \frac{x^2}{\sin x^2}$$

A. $-\infty$ B. -1 C. 0 D. 1 E. ∞

Answer is D. Well if you substitute in $x = 0$ you get zero over zero, so this is one of those where you use l'Hôpidal's rule.

$$\begin{aligned} \frac{d}{dx}x^2 &= 2x \\ \frac{d}{dx}\sin x^2 &= 2x\cos x^2 \end{aligned} \tag{15}$$

so

$$\lim_{x \to 0} \frac{x^2}{\sin x^2} = \lim_{x \to 0} \frac{2x}{\sin x^2} = \lim_{x \to 0} \frac{1}{\cos x^2} = 1 \tag{16}$$