Brief notes for 123 lectures after Christmas: 5

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In this lecture I will discuss exponential growth, first by revising the original limit of continual compounding derivation I gave before and then by introducing a differential equation for growth.

The limit of continual compounding.

If you put money in a savings account you are given interest, calculated at the end of year as a percentage of amount of money in the account.[1] If the interest rate is r then, if you started with P_0 at the start of the year, at the end of the year you are given rP_0 interest. After that you will have

$$P_1 = (1 + r)P_0 \tag{1}$$

in the account. At the end of the next year, you get $r(1+r)P_0$ interest and your total will be the sum of the money you already had and your interest:

$$P_2 = (1 + r)P_0 + r(1 + r)P_0 = (1 + r)^2 P_0. \tag{2}$$

In fact, it is easy enough to see that after t years you will have

$$P_t = (1 + r)^t P_0 \tag{3}$$

in the account.

Now, the interesting thing is that the answer would be different if your interest was added more frequently. Let's consider what happens if, instead of giving you r interest once a year, the bank gives you $r/2$ interest twice a year. If that were the case, after six months you'd have

$$P_{1/2} = \left(1 + \frac{r}{2}\right) P_0 \tag{4}$$

and after a year

$$P_1 = \left(1 + \frac{r}{2}\right)^2 P_0. \tag{5}$$

Since

$$\left(1 + \frac{r}{2}\right)^2 = 1 + r + \frac{r^2}{4} > 1 + r. \tag{6}$$

Thus, if the interest is added every six months instead of every year, you end up with more money. This is because the interest you are given after six months starts earning interest and the $r^2/4$ is the interest on that interest. If you are paid a rate of $r/2$ interest added every six months, after t years your amount would have grown to

$$P_t = \left(1 + \frac{r}{2}\right)^{2t} P_0 \tag{7}$$

[1]Well, a percentage of the *average* amount in the account, we ignore this boring complication and assume you put some money in at the start of the year and leave it from then on without touching it.

Now, say you were paid $r/3$ interest three times a year, then, after t years you would have

$$P_t = \left(1+\frac{r}{3}\right)^{3t} P_0 \tag{8}$$

or, if you are paid $r/4$ interest four times a year, after t years

$$P_t = \left(1+\frac{r}{4}\right)^{4t} P_0. \tag{9}$$

In fact, if you are paid r/n interest n times a year, after t years

$$P_t = \left(1+\frac{r}{n}\right)^{nt} P_0. \tag{10}$$

The point is that the amount gained on any given pound is still r but the amount is increasing faster if the interest is added more often and so the interest starts earning interest sooner.

Thus, the more often interest is added, the faster your amount grows. However, there is a limit and this limit is exponential growth. We define a new number

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n})^n\right) \tag{11}$$

and this means that

$$e^{rt} = \lim_{n\to\infty}\left(1+\frac{r}{n})^{nt}\right) \tag{12}$$

Thus, if growth occurs continually, as happens with bacteria, rather than annually, as happens with money,

$$P_t = e^{rt}P_0 \tag{13}$$

In Fig. 1 shows the difference between annual, quarterly and continual compounding at ten per cent interest over ten years.

In other words if a population grows so that any member has an r chance of producing a new member during a unit of time, then the entire population grows according the the exponential growth curve. This curve takes into account the fact that new members of the population are able to add to it.

Differentiating the exponential

Think about population growth has lead us to define the exponential function:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{14}$$

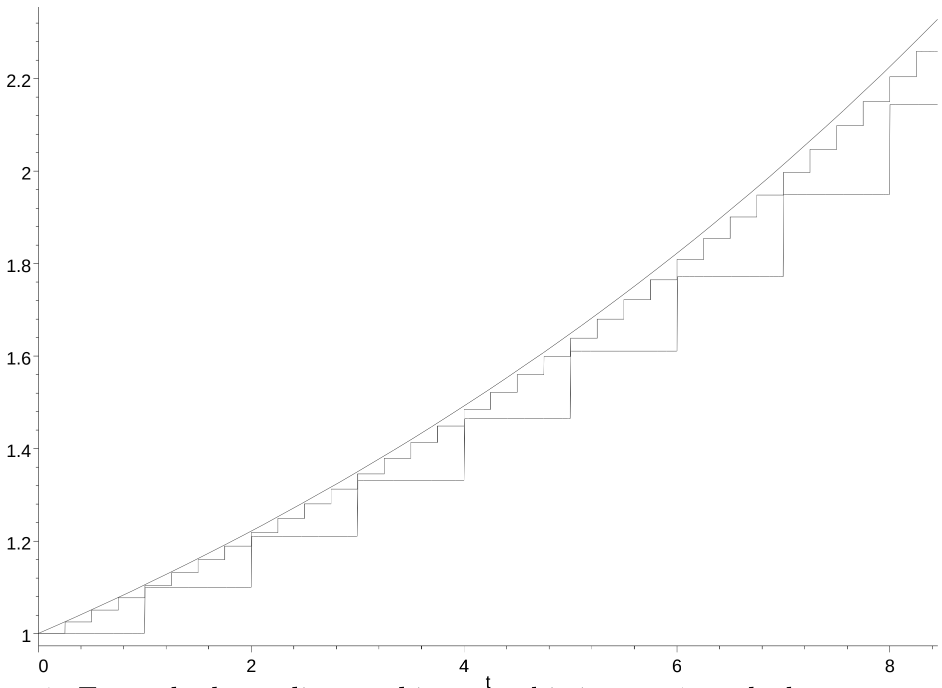

Figure 1: From the lower line working up, this is $r = .1$ worked out once a year, four times a year and continually.

On thing to note about this function is its derivative. Without worrying too much about moving the derivative through the limit we have

$$\begin{aligned}\frac{d}{dx}e^x &= \frac{d}{dx}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{n\to\infty}\frac{d}{dx}\left(1+\frac{x}{n}\right)^n \\ &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1} = e^x \qquad (15)\end{aligned}$$

Thus,

$$\frac{d}{dx}e^x = e^x, \qquad (16)$$

the derivative of the exponential is the exponential.[2]

[2] In fact, this property defines the exponential. In short, the exponential is the function whose derivative is itself. One way to see this is to think about the Taylor expansion, taking the derivative of the Taylor expansion gives back the same expansion, going the other way, if you write down a series that has the property the derivative of the series is the same series you get the Taylor expansion of the exponential.

A differential equation for growth.

This interesting property of the exponential leads us to a more sophisticated derivation of the exponential growth curve. Say a member of a population has a r chance of producing a new member in a time unit. This means that the rate of chance of the population is rP where P is the population. The derivative is the rate of change, so, the population satisfies a **differential equation**:

$$\frac{dP}{dt} = rP. \tag{17}$$

This is called a differential equation because it is an equation for P which involves derivatives. It says that the population grows at a rate proportional to the population.

It is very common for the application of mathematics to science to produce differential equations. There are many methods for solving them, we won't go into them now, we'll just observe that

$$P = P_0 e^{rt} \tag{18}$$

solve the equation. This is because

$$\frac{d}{dt}P = \frac{d}{dt}\left(P_0 e^{rt}\right) = rP_0 e^{rt} = rP \tag{19}$$

and by substituting in $t = 0$ we see that P_0 is the starting value as before. Thus, we have solved the differential equation and the solution is the growth curve.