Brief notes for 123 lectures after Christmas: 4

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The binomial expansion

You might remember the binomial expansion

$$(x+y)^n = x^n + nx^{n-1}y + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2}y^2 + \ldots + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2}y^2 + \ldots + y^n \qquad (1)$$

We derived this by thinking about what would happen if we multiplied out $(x+y)^n$. The point is that the function $(1+x)^n$ can be rewritten as a series, this is the binomial expansion. It is not an approximation, it is just what you get if you multiply out the expression. However, it is also possible to work out the Taylor expansion of the function, that also gives a series expansion for the function. Of course, it turns out that these expansions are the same. In other words, if a function is really a finite series then the Taylor series doesn't go on for ever, it just gives the finite series.

To prove this, we will work out the Taylor expansion of $(1+x)^n$. In other words, let us Taylor expand $f(x) = (1+x)^n$ about $x = 0$.[1] Now, the Taylor expansion is

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots + \frac{1}{r!}f^{(r)}x^r + \ldots \qquad (2)$$

and so we need to work out the derivatives of $f(x) = (1+x)^n$. Here goes;

$$\begin{aligned}
f'(x) &= n(1+x)^{n-1} \\
f''(x) &= \frac{d}{dx}\left[n(1+x)^{n-1}\right] = n(n-1)(1+x)^{n-2} \\
f'''(x) &= \frac{d}{dx}\left[n(1+x)^{n-1}\right] = n(n-1)(n-2)(1+x)^{n-3} \\
&\vdots \qquad (3)\\
f^{(r)}(x) &= n(n-1)(n-2)\ldots(n+1-r)(1+x)^{n-r} \\
&\vdots \qquad (4)\\
f^{(n)}(x) &= n(n-1)(n-2)\ldots 1 = n! \\
f^{(n+1)}(x) &= 0 \qquad (5)
\end{aligned}$$

So, we the Taylor series eventually runs out, there is only a finite number of terms. Next we substitute $x = 0$:

$$f'(0) = n$$

[1] As usual, since we are expanding about $x = 0$ we won't bother introducing h, that is, we don't bother writing $x = 0 + h$, we'll just use x. The policy is to avoid introducing a symbol that means exactly the same thing as a symbol you are already using.

$$
\begin{aligned}
f''(0) &= n(n-1) \\
f'''(0) &= n(n-1)(n-2) \\
&\vdots \qquad\qquad (6)\\
f^{(r)}(0) &= n(n-1)(n-2)\ldots(n+1-r) \\
&\vdots \qquad\qquad (7)\\
f^{(n)}(0) &= n(n-1)(n-2)\ldots 1 = n! \qquad (8)
\end{aligned}
$$

and we substitute that into the expansion to get

$$
\begin{aligned}
(1+x)^n &= 1 + nx + \frac{1}{2}n(n-1)x^2 + \frac{1}{6}n(n-1)(n-2)x^3 + \\
&+ \ldots + \frac{1}{r!}n(n-1)(n-2)\ldots(n+1-r)x^{n-r} + \ldots + \frac{1}{n!}n!x^n \\
&= 1 + nx + \left(\begin{array}{c} n \\ 2 \end{array}\right)x^2 + \left(\begin{array}{c} n \\ 3 \end{array}\right)x^3 + \ldots + \left(\begin{array}{c} n \\ r \end{array}\right)x^{n-r} + \ldots + x^n \qquad (9)
\end{aligned}
$$

which is the same as what is given by the binomial expansion.

It is interesting to note that this expansion for $(1+x)^n$ works even when n isn't an integer. The binomial expansion is derived for n an integer and is a finite series. However, the instructions for forming the binomial expansion can be applied more generally and even then they give the binomial expansion. As an example of this consider $\sqrt{1+x}$. The Taylor expansion around $x=0$ is

$$
\sqrt{1+x} = 1 + \frac{d}{dx}\sqrt{1+x}\bigg|_{x=0} x + \frac{1}{2}\frac{d^2}{dx^2}\sqrt{1+x}\bigg|_{x=0} x^2 + \frac{1}{6}\frac{d^3}{dx^3}\sqrt{1+x}\bigg|_{x=0} x^3 + O(x^4) \qquad (10)
$$

and, in the usual way

$$
\begin{aligned}
\frac{d}{dx}\sqrt{1+x} &= \frac{1}{2\sqrt{1+x}} \\
\frac{d^2}{dx^2}\sqrt{1+x} &= -\frac{1}{4(1+x)^{3/2}} \\
\frac{d^3}{dx^3}\sqrt{1+x} &= \frac{3}{8(1+x)^{5/2}} \qquad (11)
\end{aligned}
$$

Thus, substituting in $x=0$ gives

$$
\begin{aligned}
\frac{d}{dx}\sqrt{1+x}\bigg|_{x=0} &= \frac{1}{2} \\
\frac{d^2}{dx^2}\sqrt{1+x}\bigg|_{x=0} &= -\frac{1}{4} \\
\frac{d^3}{dx^3}\sqrt{1+x}\bigg|_{x=0} &= \frac{3}{8} \qquad (12)
\end{aligned}
$$

and so the Taylor expansion is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4) \tag{13}$$

Now, say you tried to binomial expand $\sqrt{1+x}$, you'd get

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \left(\begin{array}{c} 1/2 \\ 2 \end{array}\right) x^2 + \left(\begin{array}{c} 1/2 \\ 3 \end{array}\right) x^3 + O(x^4) \tag{14}$$

Now, when we introduced the binomial symbol it was only for integers, how, let's try and carry on regardless

$$\left(\begin{array}{c} 1/2 \\ 2 \end{array}\right) = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} = -\frac{1}{8} \tag{15}$$

and

$$\left(\begin{array}{c} 1/2 \\ 3 \end{array}\right) = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3\times 2} = \frac{1}{16}. \tag{16}$$

Substituting into the binomial expansion gives back the Taylor expansion. In other words, the binomial expansion was discussed before for n an integer, but, when n isn't an integer the formula still seems to work, except the series isn't finite. It gives the Taylor expansion.

To summarize this lecture, the binomial expansion and the Taylor expansion both give series expressions for functions. The Taylor expansion applies to far more functions[2] than the binomial expansion, however, when they both apply they both give the same answer.

Of course, this lecture has been quite theoretical, any actual question on the Taylor expansion will be of the please calculate the expansion of this around that form. What I have just said about $\sqrt{1+x}$ does give a quick way of calculating Taylor expansions for certain functions if you feel comfortable with using it.

[2]This isn't something you should worry about, but it is interesting to note that not all functions have Taylor expansions, the obvious counterexample is

$$e^{-1/x}.$$