Brief notes for 123 lectures after Christmas: 3

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More on the Taylor expansion

Here is another example, consider expanding $\log_e x$ around $x = 1$. Let $x = 1 + h$ and use the usual formula:[1]

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{3!}f'''(a)h^3 + \ldots \tag{1}$$

So, in the case we are looking at

$$\log_e(1+h) = \log_e 1 + \left.\frac{d\log_e x}{dx}\right|_{x=1} h + \frac{1}{2}\left.\frac{d^2\log_e x}{dx^2}\right|_{x=1} h^2 + \frac{1}{6}\left.\frac{d^3\log_e x}{dx^3}\right|_{x=1} h^3 + O(h^4). \tag{2}$$

Now, we are able to do these differentiations:

$$\begin{aligned} \frac{d\log_e x}{dx} &= \frac{1}{x} \\ \frac{d^2\log_e x}{dx^2} &= \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \\ \frac{d^3\log_e x}{dx^3} &= -\frac{d}{dx}\frac{1}{x^2} = \frac{2}{x^3} \end{aligned} \tag{3}$$

and so

$$\begin{aligned} \left.\frac{d\log_e x}{dx}\right|_{x=1} &= 1 \\ \left.\frac{d^2\log_e x}{dx^2}\right|_{x=1} &= -1 \\ \left.\frac{d^3\log_e x}{dx^3}\right|_{x=1} &= 2 \end{aligned} \tag{4}$$

We substitute these results, and the fact that $\log_e 1 = 0$, into (2) to get

$$\log_e(1+h) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3 + O(h^4). \tag{5}$$

In Fig. 1 we can see how well the log is approximated by this Taylor expansion.

[1] Don't get confused by the h, all we are doing is, we are looking at how the function behaves around a, so we say, let x be near a, so $x = a + h$ where h is a small number. Often when we are looking at the expansion around $x = 0$ we don't bother introducing the h and we just expand in x. In other words, the expansion around $x = 0$ is often written as

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \ldots.$$

Sometimes the expansion around $x = 0$ is called the Maclaurin expansion.

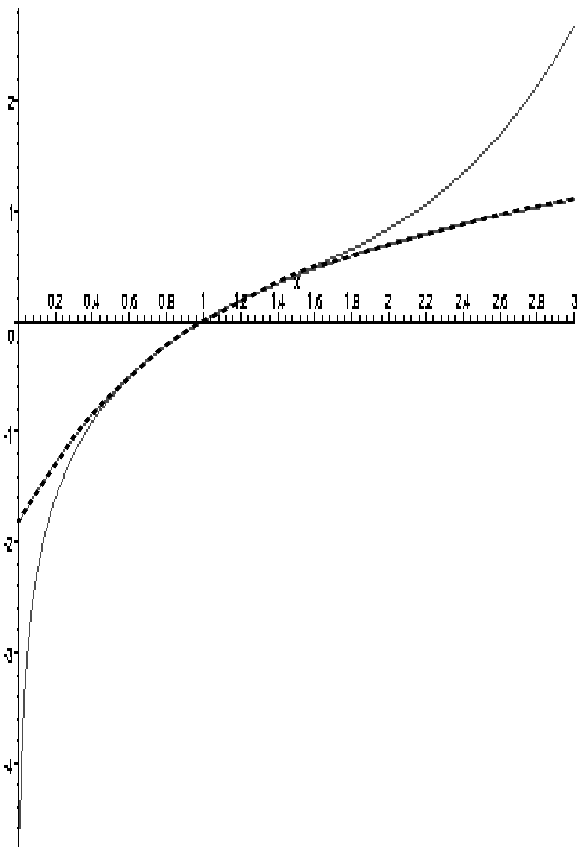

Figure 1: Approximating $\log_e x$ by the first three nonzero terms in its Taylor expansion. The dotted line is the Taylor approximation.

Summary

So, in summary, a Taylor expansion is a series approximation for a function around a point. A series is a sum of terms, in the Taylor expansion each of these terms is a number multiplying a successively higher power of some number h. Thus, a polynomial is a finite series and a very easy thing to deal with. To work out the value of a polynomial at some point, you just substitute in for x and work out the number by multiplying an adding. Most functions are not finite series. What the Taylor expansion does is it gives you a finite expansion which approximates the function. In principal, the Taylor expansion goes on for ever and often the infinite expansion converges to the function. However, in practical calculations and in most applications, we stop Taylor expanding after a number of terms and we say the function is approximate by the finite series.

If you are asked to work out a Taylor expansion of some function $f(x)$ around $x = a$, you start by writing $x = a + h$. Usually you will be told to work out the expansion to some order in h, that means that order is the power of h in the highest term you work out. Say you are asked to work out h to the third order in h, then what you want is the expansion

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(a)h^3 + O(h^4). \tag{6}$$

and to get the answer all you need to do is work out what $f(a)$ is and what $f'(a)$ is and what $f''(a)$ is and what $f'''(a)$ is. Now $f'(a)$ is the value of the differential of f at a and can also be written as

$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) \tag{7}$$

To work out it, you differentiate f with respect to a and then substitute in $x = a$. In the previous example with the log $f'(x) = 1/x$ and a was 1 so $f'(a) = 1$. In the same way, $f''(a)$ is the second derivative of f with respect to x evaluated at $x = a$. This is also written as

$$\left.\frac{d^2f}{dx^2}\right|_{x=a} = f''(a) \tag{8}$$

and to work it out you differentiate f twice with respect to a and substitute in $x = a$. Thus, in the example above, you differentiate $\log_e x$ twice to get $-1/x^2$ and substitute in $x = 1$ to get -1.

So, the Taylor expansion is a series expansion which can be used to approximate a function. The first four terms are given in equation (6) and the rth term is $(1/r!)f^{(r)}(a)h^r$.