Brief notes for 123 lectures after Christmas

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L'Hôpidal's rule

L'Hopidals rule allows us to calculate limits of fractions where both the numerator and the denominator are going to zero. We examined limits of this form before. We looked at, for example,

$$\lim_{x\to 1}\frac{x^2-1}{x^2+x-2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x+2)}=\lim_{x\to 1}\frac{x+1}{x+2}=\frac{2}{3} \tag{1}$$

In this example we cancel the $x-1$ factor above and below the line and then substitute $x=1$ into the remaining expression. If the limit sign wasn't there we could only do the cancellation when $x\neq 1$ because at $x=1$ the cancellation amounts to dividing by zero.

The same factorization and cancellation technique work for

$$\lim x\to 3\frac{x-3}{\sqrt{x-2}-1} \tag{2}$$

because[1] $x-3=(\sqrt{x-2}-1)(\sqrt{x-2}+1)$ so

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-1}=\lim_{x\to 3}\frac{(\sqrt{x-2}-1)(\sqrt{x-2}+1)}{\sqrt{x-2}-1}=\lim_{x\to 3}(\sqrt{x-2}+1)=2 \tag{3}$$

However, a more powerful and generally easier method is to use **L'Hôpidal's rule**.

L'Hôpidal's rule says that if you have a limit

$$\lim_{x\to a}\frac{f(x)}{g(x)} \tag{4}$$

where $f(a)=0$ and $g(a)=0$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{5}$$

In other words, you can differenciate above and below the line of the fraction without changing the value of the fraction.[2]

In the example we were considering $x-3$ and $\sqrt{x-2}-1$ are both zero for $x=3$ and so it is acase were l'Hôpidal's rule can be used. Now,

$$\begin{aligned}\frac{d}{dx}(x-3) &= 1\\ \frac{d}{dx}(\sqrt{x-2}-1 &= \frac{1}{2\sqrt{x-2}}\end{aligned} \tag{6}$$

[1]Recall $(a-b)(a+b)=a^2-b^2$ and in this case $a=sqrtx-2$ and $b=1$.

[2]This isn't the same as differenciating the fraction, you don't use the quotient rule. Differenciating the fraction will usually change the limit, in L'Hopidal's rule you differenciate f and seperately you differenciate g.

and so

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-1}=\lim_{x\to 3}2\sqrt{x-2}=2 \tag{7}$$

giving the same answer back.

Here is another example:

$$\lim_{x\to 0}\frac{\sin x}{x}. \tag{8}$$

Again, when $x=0$ the numerator and the demoninator are both zero so we can use the rule to give

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\cos x=1. \tag{9}$$

A similar example is

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{2}=1 \tag{10}$$

Sometimes you have to use the rule more than once. For example

$$\lim_{x\to 0}\frac{\tan^2 x}{x^2}=\lim_{x\to 0}\frac{\tan x}{x\cos^2 x} \tag{11}$$

but we still can't work this out because $\tan 0=0$ and $x\cos^2 x$ is also zero for $x=0$. Thus, the fraction is still undefined at the limit point, but we can do l'Hôpidal's rule again. Since

$$\frac{d}{dx}(x\cos^2 x)=\cos^2 x+2x\sin x\cos x \tag{12}$$

we have

$$\lim_{x\to 0}\frac{\tan x}{x\cos^2 x}=\lim_{x\to 0}\frac{1}{\cos^2 x(\cos^2 x+2x\sin x\cos x)}=1 \tag{13}$$

So you see, although applying l'Hôpidal's rule the first time wasn't enough, things do improve and applying it the second time gives the answer. It will be more obvious why this happens and why l'Hopidal's rule works, when we have studied the Taylor series.