MA3466 Tutorial Sheet 6: solutions to q1-3[1]

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1. (C& T 3.2) AEP and mutual information. Let(X_i, Y_i) be i.i.d with joint distributions $p(x, y)$. We form the log likelihood ration of the hypothesis that X and Y are independent versus the hypothesis that they are dependent. What is the limit of

$$\frac{1}{n}\log\frac{p(\mathbf{X})p(\mathbf{Y})}{p(\mathbf{X},\mathbf{Y})} \tag{1}$$

*Solution:*This is done in a way that mimics the original proof of the AEP; first we use independence

$$\frac{1}{n}\log\frac{p(\mathbf{X})p(\mathbf{Y})}{p(\mathbf{X},\mathbf{Y})} = \frac{1}{n}\log\frac{\prod_i p(X_i)p(Y_i)}{\prod_i p(X_i,Y_i)} = \frac{1}{n}\sum_i \log\frac{p(X_i)p(Y_i)}{p(X_i,Y_i)} \tag{2}$$

Now, if the X_i and the Y_i are i.i.d. then is the expression in the sum; so, in the sense of the law of the large numbers

$$\frac{1}{n}\log\frac{p(\mathbf{X})p(\mathbf{Y})}{p(\mathbf{X},\mathbf{Y})} \to E\log\frac{p(X)p(Y)}{p(X,Y)} = -I(X,Y) \tag{3}$$

2. (C& T 3.3) A piece of cake. A cake is sliced roughly in half and the largest piece selected each time, the other bits being discarded. Assume $p(2/3, 1/3) = 3/4$ and $p(2/5, 3/5) = 1/4$. How large, to the first order in the exponent, is the piece of cake after n cuts.

*Solution:*So here if there are b 2/3 cuts and s 3/5 cuts the size of the remaining piece will be

$$S = \left(\frac{2}{3}\right)^b\left(\frac{3}{5}\right)^s \approx \left(\frac{2}{3}\right)^{3n/4}\left(\frac{3}{5}\right)^{n/4} \tag{4}$$

to leading order.

3. (C& T 3.6) AEP-like limit. Let X_1, X_2 and so on be i.i.d., drawn with distribution $p(x)$, what is

$$\lim_{n\to\infty}[p(X_1, X_2, \ldots, X_n)]^{1/n} \tag{5}$$

For this you need to know the strong law of large numbers: to prove the AEP we used the weak law:

$$\frac{1}{n}\sum X_i \to EX \tag{6}$$

in probability, the strong law states that it approaches it almost surely.

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*Solution:*So the trick here is to use a log so as to be able to change a product into a sum

$$\begin{aligned}\lim_{n\to\infty}[p(X_1, X_2, \ldots, X_n)]^{1/n} &= \lim_{n\to\infty} 2^{\log[p(X_1,X_2,\ldots,X_n)]^{1/n}} \\ &= \lim_{n\to\infty} 2^{\frac{1}{n}\sum\log p(X_i)} = 2^{E\log p(X)} = 2^{-H(X)}\end{aligned} \tag{7}$$