472 Tutorial Sheet 2 Solutions to q2: information audit[1]

11 Febuary 2008

Here I go through the solution to the 12 coin problem and try and track the information

(C&T 2.7) Suppose that one has n coins, among which there is one counterfeit coin. If there is a counterfeit coin it will weight either less or more than the other coins. The coins are weighed using a balance.

1. Find an upper bound on the number of coins n so that k weighings will find the counterfeit coin, if any, and correctly declare it to be heavier or lighter.

2. What is the coin-weighing strategy for $k = 3$ weighings and 12 coins?

Soln

Here I consider the second part. Lets start by numbering the coins from one to 12. The three weighings are called Y_1, Y_2 and Y_3 and the outcomes are named l, r and n for left heavy, right heavy and neither heavy. I follow the strategy described in the PS2 solutions.

The first weighing Y_1

We weigh $\{1, 2, 3, 4\}$ versus $\{5, 6, 7, 8\}$; the three outcomes are equally likely so

$$p_{Y_1}(l) = p_{Y_1}(r) = p_{Y_1}(n) = 1/3 \tag{1}$$

and $H(Y_1) = \log 3$.

$Y_1 = l$ or $Y_1 = r$

In this case we weigh $\{1, 2, 5\}$ versus $\{3, 4, 6\}$ leaving $\{7, 8\}$ aside. Since the bent coin is one of 1 to 8 and only 7 and 8 are left aside

$$\begin{aligned} p_{Y_2|Y_1}(n|l) &= 1/4 \\ p_{Y_2|Y_1}(l|l) = p_{Y_2|Y_1}(r|l) &= 3/8 \end{aligned} \tag{2}$$

and

$$\begin{aligned} H(Y_2|Y_1 = l) &= -\frac{1}{4}\log\frac{1}{4} - 2\frac{3}{8}\log\frac{3}{8} \\ &= \frac{11}{4} - \frac{3}{4}\log 3 \end{aligned} \tag{3}$$

$H(Y_2|Y_1 = r)$ is the same.

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/472`

What about $Y_1 = n$?

In this case we weigh $\{9, 10\}$ versus $\{11, 1\}$; again, the bent coin must be one of 9 to 12 and only 12 is left aside. Hence

$$\begin{aligned} p_{Y_2|Y_1}(n|n) &= 1/4 \\ p_{Y_2|Y_1}(l|n) = p_{Y_2|Y_1}(r|n) &= 3/8 \end{aligned} \tag{4}$$

and, again

$$\begin{aligned} H(Y_2|Y_1 = n) &= -\frac{1}{4}\log\frac{1}{4} - 2\frac{3}{8}\log\frac{3}{8} \\ &= \frac{11}{4} - \frac{3}{4}\log 3 \end{aligned} \tag{5}$$

The conditional entropy $H(Y_2|Y_1)$

Now

$$H(Y_2|Y_1) = p_{Y_1}(l)H(Y_2|l) + p_{Y_1}(r)H(Y_2|r) + p_{Y_1}(r)H(Y_2|r) = \frac{11}{4} - \frac{3}{4}\log 3 \tag{6}$$

In fact this is approximately 1.56, $\log 3 \approx 1.59$, so, although

$$H(Y_2, Y_1) = H(Y_1) + H(Y_2|Y_1) < 2H(Y_1) \tag{7}$$

the difference is only about .03 bits.

$Y_1 = l$ and $Y_2 = l$ and similar

If $Y_1 = l$ and $Y_2 = l$ then for Y_3 we weigh 1 against 2 knowing that the bent coin is 1, 2 or 6. Each is equally likely, so

$$p_{Y_3|Y_2,Y_1}(l|l,l) = p_{Y_3|Y_2,Y_1}(r|l,l) = p_{Y_3|Y_2,Y_1}(n|l,l) = \frac{1}{3} \tag{8}$$

and

$$H(Y_3|Y_2 = l, Y_1 = l) = \log 3 \tag{9}$$

You get the same for other combinations of Y_1 and Y_2 each being l or r.

$Y_1 = l$ and $Y_2 = n$ and similar

If $Y_1 = l$ and $Y_2 = n$ then we know 7 or 8 is bent and we weigh them against each other, the result cannot be n so

$$\begin{aligned} p_{Y_3|Y_2,Y_1}(l|n,l) = p_{Y_3|Y_2,Y_1}(r|n,l) &= \frac{1}{2} \\ p_{Y_3|Y_2,Y_1}(n|n,l) &= 0 \end{aligned} \tag{10}$$

and

$$H(Y_3|Y_2 = n, Y_1 = l) = \log 2 = 1 \tag{11}$$

Y_3 when $Y_1 = n$

This actually works in a similar way to the $Y_1 = l$ cases: if $Y_2 = l$ we know the coin is either 10 or 11 heavy or it is 12 and light: this is the same as the $(Y_1 = l, Y_2 = l)$ case. If $Y_2 = n$ we have only 12 left and we weigh it against the coin 1 which is known to not be bent; here there is no possibility of $Y_3 = n$, like the $(Y_1 = l, Y_2 = n)$ case.

The joint distribution $p_{Y_2,Y_1}(y_2, y_1)$

To work out $H(Y_3|Y_2, Y_2)$ we need the joint distribution for Y_1 and Y_2; we can work that out from what we have above; for example

$$\begin{aligned} p_{Y_2,Y_1}(l,l) &= p_{Y_2|Y_1}(l)p_{Y_1}(l) = \frac{3}{8}\frac{1}{3} = \frac{1}{8} \\ p_{Y_2,Y_1}(r,l) &= p_{Y_2|Y_1}(r)p_{Y_1}(l) = \frac{3}{8}\frac{1}{3} = \frac{1}{8} \\ p_{Y_2,Y_1}(n,l) &= p_{Y_2|Y_1}(n)p_{Y_1}(l) = \frac{1}{4}\frac{1}{3} = \frac{1}{12} \end{aligned} \tag{12}$$

and, in fact, the same pattern is found for the $Y_1 = r$ and the $Y_1 = n$ probabilities.

The conditional entropy $H(Y_3|Y_2, Y_1)$

Well this is a long sum over all the possibilities but it is actually quite easy, for $Y_1 = l$ for example, we have $Y_2 = l$ or $Y_2 = r$ each with probability 1/8 with corresponding

$$H(Y_3|Y_2 = l, Y_1 = l) = H(Y_3|Y_2 = r, Y_1 = l) = \log 3 \tag{13}$$

and $Y_2 = n$ with probability 1/12 and

$$H(Y_3|Y_2 = n, Y_1 = l) = 1 \tag{14}$$

The same repeats for $Y_1 = r$ and, less obviously, for $Y_1 = b$. Hence

$$\begin{aligned} H(Y_3|Y_2, Y_1) &= \sum p_{Y_2,Y_1}(y_2, y_1)H(Y_3|Y_2 = y_2, Y_1 = y_1) \\ &= 3\left(2\frac{1}{8}\log 3 + \frac{1}{12}\right) = \frac{1}{4} + \frac{3}{4}\log 3 \end{aligned} \tag{15}$$

The punchline: $H(Y_3, Y_2, Y_1)$

So

$$\begin{aligned} H(Y_3, Y_2, Y_1) &= H(Y_3|Y_2, Y_1) + H(Y_2|Y_1) + H(Y_1) \\ &= \frac{1}{4} + \frac{3}{4}\log 3 + \frac{11}{4} - \frac{3}{4}\log 3 + \log 3 \\ &= 3 + \log 3 \end{aligned} \tag{16}$$

However, we also have $H(X) = \log 2n$ where n is the number of coins. Hence

$$H(X) = \log 24 = \log 2^3 3 = 3 + \log 3 = H(Y_3, Y_2, Y_1) \tag{17}$$

Which is very cool.