MA3466 Tutorial Sheet 4[1]

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1. (C&T 2.16) Bottleneck. Given a Markov chain $X_1 \to X_2 \to X_3$ where the sets of outcomes have cardinalities $|\mathcal{X}_1| = n$, $|\mathcal{X}_2| = k$ and $|\mathcal{X}_3| = m$ with k is less than both m and n.

   (a) Show that the dependence of X_1 and X_3 is limited by the bottleneck by proving that $I(X_1; X_3) \leq \log k$.

   (b) Evaluate $I(X_1; X_3)$ for $k = 1$ and conclude that no dependence can survive such a bottleneck.

2. Consider the game of &$^{\text{TM}}$: a board game with numbered squares, the idea being to race your token along the square by flipping a coin, heads you advance one place, tails you stay put. Let Y_1, Y_2 and Y_3 be you position after one, two and three turns. Clearly $Y_1 \to Y_2 \to Y_3$, show this by calculating the conditional probabilites and showing

$$p(x, y, z) = p(x)p(x|y)p(z|y) \tag{1}$$

   Calculate $I(Y_1; Y_2)$ and $I(Y_1; Y_3)$.

3. (C&T 2.43b) A fair six-sided dice is rolled. What is the mutual information between the top face of the dice and the side most facing you?

4. (C&T 2.29) For random variables X, Y and Z prove the following inequalities and find conditions for equality.

   (a) $H(X, Y|Z) \geq H(X|Z)$

   (b) $I(X, Y; Z) \geq I(X; Z)$

   (c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$

   (d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$

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