MA22S3 ODE work sheet - outline solutions questions 4-7.[1]

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8 Find the general solution for $\ddot{y} - 5\dot{y} + 6y = e^{3t}$, what is the solution is $y(0) = 4$ and $\dot{y}(0) = 1$.

Solution: So in this case we already know the complementary function, it is

$$y_c = C_1 e^{3t} + C_2 e^{2t} \tag{1}$$

To find the particular function we first notice that the exponential on the right hand side, $\exp 3t$, already appears in the complementary function, so the appropriate ansatz is $y = Ct \exp 3t$. Now

$$\begin{aligned} \dot{y} &= Ce^{3t} + 3Cte^{3t} \\ \ddot{y} &= 6Ce^{3t} + 9Cte^{3t} \end{aligned} \tag{2}$$

Substituting back into the differential equation and cancelling the $\exp 3t$ we get

$$6C + 9Ct - 5C - 15Ct + 6Ct = 1 \tag{3}$$

of $C = 1$, hence the general solution is

$$y = te^{3t} + C_1 e^{3t} + C_2 e^{2t} \tag{4}$$

Now

$$y(0) = C_1 + C_2 = 4 \tag{5}$$

and

$$\dot{y} = 1 + 3C_1 + 2C_2 = 1 \tag{6}$$

so $C_1 = -8$ and $C_2 = 12$ and

$$y = te^{3t} - 8e^{3t} + 12e^{2t} \tag{7}$$

9 Find the general solution for $\ddot{y} - 6\dot{y} + 9y = e^{3t}$, what is the solution is $y(0) = 4$ and $\dot{y}(0) = 1$.

Solution: This time we do need to calculate the complementary function, the auxiliary equation is

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \tag{8}$$

so the complementary function is

$$y_c = C_1 e^{3t} + C_2 te^{3t} \tag{9}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3

Now, the trouble is that this means that the ansatz for the particular solution has to be $y = Ct^2 \exp 3t$ so

$$\dot{y} = 2Cte^{3t} + 3Ct^2e^{3t} \tag{10}$$

and

$$\ddot{y} = 12Cte^{3t} + 2Ce^{3t} + 9Ct^2e^{3t} \tag{11}$$

and so, substituting in and cancelling the $\exp 3t$

$$12Ct + 2C + 9Ct^2 - 12Ct - 18C + 9Ct^2 = 1 \tag{12}$$

so$C = -1/16$. Now

$$y = -\frac{1}{16}t^2e^{3t} + C_1e^{3t} + C_2te^{3t} \tag{13}$$

so

$$y(0) = C_1 = 4 \tag{14}$$

and

$$\dot{y}(0) = C_1 + C_2 = 1 \tag{15}$$

so $C_2 = -3$ and

$$y = -\frac{1}{16}t^2e^{3t} + 4e^{3t} - 3te^{3t} \tag{16}$$