MA22S3 ODE work sheet - outline solutions questions 4-7.[1]

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Questions: lots of ODE questions for practising.

4 Find the general solution for

(a) $\ddot{y} + 4\dot{y} + 4y = 0$

(b) $\ddot{y} + 10\dot{y} + 25y = 0$

(c) $\ddot{y} + 8\dot{y} + 16y = 0$

In each example, what is the solution if $y(0) = 2$ and $\dot{y}(0) = -3$.

Solution: So first of course you substitute $y = \exp(\lambda t)$ and, after cancelling across the exponentials you get

$$\lambda^2 + 4\lambda + 4 = 0 \tag{1}$$

Factorizing gives

$$(\lambda + 2)^2 = 0 \tag{2}$$

and so there is only one λ and so the solution is

$$y = C_1 e^{-2t} + C_2 t e^{-2t} \tag{3}$$

Since we will need it for the initial conditions lets do $\dot{y}$:

$$\dot{y} = -2C_1 e^{-2t} - 2C_2 t^{-2t} + C_2 e^{-2t} \tag{4}$$

Now $y(0) = C_1 = 2$ by the initial conditions, and $\dot{y}(0) = -2C_1 + C_2 = -4 + C_2 = -3$ so $C_2 = 1$ and

$$y = 2e^{-2t} + te^{-2t} \tag{5}$$

The other two are very similar, in the next case we have

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0 \tag{6}$$

so

$$y = C_1 e^{-5t} + C_2 t e^{-5t} \tag{7}$$

and so

$$\dot{y} = -5C_1 e^{-5t} - 5C_2 t^{-5t} + C_2 e^{-5t} \tag{8}$$

Again $y(0) = C_1 = 2$ and $\dot{y}(0) = -5C_1 + C_2 = -10 + C_2 = -3$ so $C_2 = 7$ and

$$y = 2e^{-5t} + 7te^{-5t} \tag{9}$$

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Finally, for the last one

$$\lambda^2 + 8\lambda + 16 = (\lambda + 4)^2 = 0 \tag{10}$$

so

$$y = C_1 e^{-4t} + C_2 t e^{-4t} \tag{11}$$

and so

$$\dot{y} = -4C_1 e^{-4t} - 4C_2 t^{-4t} + C_2 e^{-4t} \tag{12}$$

Again $y(0) = C_1 = 2$ and $\dot{y}(0) = -4C_1 + C_2 = -8 + C_2 = -3$ so $C_2 = 5$ and

$$y = 2e^{-4t} + 5te^{-4t} \tag{13}$$

5 Find the general solution for $\ddot{y} - 2\dot{y} + 10y = 0$.

Solution: So start as always by substituting $y = \exp(\lambda t)$ giving

$$\lambda^2 - 2\lambda + 10 = 0 \tag{14}$$

Hence

$$\lambda = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i \tag{15}$$

This means

$$y = C_1 e^{(1+3i)t} + C_2 e^{(1-3i)t} \tag{16}$$

Lets try and make this explicitly real

$$\begin{aligned} y &= e^t[C_1(\cos 3t + i \sin 3t) + C_2(\cos 3t - i \sin 3t)] \\ &= e^t[(C_1 + C_2)\cos 3t + i(C_1 - C_2)\sin 3t] \end{aligned} \tag{17}$$

hence the solution is real provided $C_1 = C_2^*$, letting $A = C_1 + C_2$ and $B = i(C_1 - C_2)$ we get

$$y = [A \cos 3t + B \sin 3t]e^t. \tag{18}$$

6 Find the general solution for $\ddot{y} - 5\dot{y} + 6y = 25$, what is the solution is $y(0) = 4$ and $\dot{y}(0) = 1$.

Solution: So to solve an inhomogeneous problem first solve the homogeneous problem to find the complementary function:

$$\ddot{y}_c - 5\dot{y}_c + 6y_c = 0 \tag{19}$$

using the usual ansatz method

$$y_c = C_1 e^{3t} + C_2 e^{2t} \tag{20}$$

Next we need a particular function, that is, at least one solution to the full differential equation. The right hand side is a constant, the usual rule is that if the right hand side is an exponential, substitute the same exponential; one example of this is the right hand side a constant, that's like having an exponential with a zero exponent, in other words try $y = C$, subing in we get

$$6C = 25 \tag{21}$$

hence $C = 25/6$ and, adding the complementary function

$$y = \frac{25}{6} + C_1 e^{3t} + C_2 e^{2t} \tag{22}$$

Now we just need to fix the constants from the initial conditions, from $y(0) = 4$

$$4 = \frac{25}{6} + C_1 + C_2 \tag{23}$$

and from $\dot{y}(0) = 1$

$$1 = 3C_1 + 2C_2 \tag{24}$$

giving $C_1 = 4/3$ and $C_2 = -3/2$ so

$$y = \frac{25}{6} + \frac{4}{3}e^{3t} - \frac{3}{2}e^{2t} \tag{25}$$

7 Find the general solution for $\ddot{y} - 5\dot{y} + 6y = e^{-3t}$, what is the solution is $y(0) = 4$ and $\dot{y}(0) = 1$.

Solution: So in this case we already know the complementary function, it is

$$y_c = C_1 e^{3t} + C_2 e^{2t} \tag{26}$$

To find the particular function we substitute $y = C \exp -3t$ since $\exp -3t$ is the exponential on the right hand side. This gives

$$9C + 15C + 6C = 1 \tag{27}$$

or $C = 1/30$. Thus

$$y = \frac{1}{30}e^{-3t} + C_1 e^{3t} + C_2 e^{2t} \tag{28}$$

Now the initial conditions give

$$4 = \frac{1}{30} + C_1 + C_2 \tag{29}$$

and from $\dot{y}(0) = 1$

$$1 = -\frac{1}{10} + 3C_1 + 2C_2 \tag{30}$$

so $C_1 = -41/6$ and $C_2 = 54/5$ and

$$y = \frac{1}{30}e^{-3t} - \frac{41}{6}e^{3t} + \frac{54}{5}e^{2t} \tag{31}$$