MA22S3 ODE work sheet - outline solutions questions 10-11.[1]

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8 Find the general solution for $\ddot{y} - 5\dot{y} + 6y = 2\sin 4t$, what is the solution is $y(0) = 4$ and $\dot{y}(0) = 1$.

Solution: So in this case we already know the complementary function, it is

$$y_c = C_1 e^{3t} + C_2 e^{2t} \tag{1}$$

Before looking for the particular function we should expand the $2\sin 4t = -ie^{4it} + ie^{-4it}$. The easiest thing is to do each separately, first

$$\ddot{y} - 5\dot{y} + 6y = -ie^{4it} \tag{2}$$

Substitute $y = C\exp 4it$, so

$$-16C - 20iC + 6C = -i \tag{3}$$

and

$$C = \frac{i}{10+20i} = \frac{1}{10}\frac{i}{1+2i} \tag{4}$$

Similarly for

$$\ddot{y} - 5\dot{y} + 6y = ie^{-4it} \tag{5}$$

Substitute $y = C\exp -4it$, so

$$-16C + 20iC + 6C = i \tag{6}$$

so

$$C = -\frac{i}{10-20i} = -\frac{1}{10}\frac{i}{1-2i} \tag{7}$$

Hence

$$y = \frac{1}{10}\left(\frac{i}{1+2i}e^{4it} - \frac{i}{1-2i}e^{-4it}\right) + C_1 e^{3t} + C_2 e^{2t} \tag{8}$$

To put this into explicitly real form you need to rewrite the complex fractions so the complex number is no longer on the denominator, for example

$$\frac{i}{1+2i} = \frac{i}{1+2i}\frac{1-2i}{1-2i} = \frac{2+i}{5} \tag{9}$$

and to expand out the complex exponentials

$$e^{4it} = \cos 4t + i\sin 4t \tag{10}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

Putting all this together we have

$$y = \frac{1}{5}\left(\frac{2}{5}\cos 4t - \frac{1}{5}\sin 4t\right) + C_1 e^{3t} + C_2 e^{2t} \tag{11}$$

Working out the constants gives

$$y(0) = \frac{2}{25} + C_1 + C_2 = 4 \tag{12}$$

and second

$$\dot{y}(0) = -\frac{4}{25} + 3C_1 + 2C_2 = 1 \tag{13}$$

Solving these gives $C_1 = -167/25$ and $C_2 = 53/5$.

11 Find the general solution for $\ddot{y} + 6\dot{y} + 10y = 2\sin 2t$.

Solution: So here the complementary function is

$$y_c = C_1 e^{-3t+it} + C_2 e^{-3t-it} = (A\sin t + B\cos t)e^{-3t} \tag{14}$$

so that's pretty annoying. Now we need to solve for two particular functions. For

$$\ddot{y} + 6\dot{y} + 10y = -ie^{2it} \tag{15}$$

use $y = C\exp 2it$ giving

$$(-4 + 12i + 10)C = -i \tag{16}$$

so

$$C = -\frac{i}{6+12i} \tag{17}$$

The other particular solution has to be the conjugate of this one, since the answer has to be real, so

$$y = -\frac{i}{6+12i}e^{2it} + \frac{i}{6-12i}e^{-2it} + (A\sin t + B\cos t)e^{-3t} \tag{18}$$

Simplifying this by putting $\exp 2it = \cos 2t + i\sin 2t$ would give

$$y = -\frac{1}{15}cos2t + \frac{2}{15}\sin 2t + (A\sin t + B\cos t)e^{-3t} \tag{19}$$