MA22S3 ODE work sheet, outline solns for q1-3.[1]

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1. Find the general solution for

   (a) $t\dot{y} + y = t^4$

   (b) $(1 - t^2)\dot{y} - ty = 1$

   (c) $(t - 2)\dot{y} - y = (t - 2)^3$

   In the third example, what is the solution if $y = 10$ when $t = 4$.

   Solution: In the first it is easy to see the left hand side is

$$\frac{d}{dt}(ty) = t^4 \tag{1}$$

so integrating both sides give

$$ty = \frac{1}{4}t^5 + C \tag{2}$$

so

$$y = \frac{1}{4}t^5 + \frac{C}{t} \tag{3}$$

The second one can't be done by staring at the left hand side, divide across by $(1-t^2)$ so

$$\dot{y} - \frac{t}{1-t^2}y = \frac{1}{1-t^2} \tag{4}$$

Now, the integrating factor is

$$\lambda = e^{\int_0^t p(t'} dt' \tag{5}$$

where $p(t) = -t/(1 - t^2)$, now

$$-\int_0^t \frac{t}{1-t'^2}dt' = \frac{1}{2}\ln(1-t^2) \tag{6}$$

and so

$$\lambda = \sqrt{1-t^2} \tag{7}$$

and

$$\frac{d}{dt}(\sqrt{1-t^2}y) = \frac{1}{\sqrt{1-t^2}} \tag{8}$$

and integrating both sides gives

$$\sqrt{1-t^2}y = \sin^{-1} t + c \tag{9}$$

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The third example, again, divide across so that the derivative term has no a coefficient of one

$$\dot{y} - \frac{1}{t-2}y = (t-2)^2 \tag{10}$$

and

$$-\int_0^t \frac{1}{t'-2}dt' = -\ln{(t-2)} \tag{11}$$

so the integrating factor is $1/(t-2)$ and we have

$$\frac{d}{dt}\left(\frac{y}{t-2}\right) = t-2 \tag{12}$$

Integrating both sides

$$y = \frac{1}{2}(t-2)^3 + C(t-2) \tag{13}$$

Substituting in $t = 4$ and solving for C gives $C = 3$ and hence

$$y = \frac{1}{2}(t-2)^3 + 3(t-2) \tag{14}$$

2. Bernoulli Equations: a Bernoulli equation is an equation of the form

$$\dot{y} + py = qy^n \tag{15}$$

where p and q are functions of t. By dividing both sides by y^n and rewriting in terms of $z = y^{1-n}$ reduce the Bernoulli equation to a first order linear inhomogeneous differential equation for z.

Solution: So if we divide both sides by y^n we get

$$y^{-n}\dot{y} + py^{1-n} = q \tag{16}$$

Now

$$\frac{d}{dt}y^(1-n) = (1-n)y^{-n}\frac{d}{dt}y \tag{17}$$

hence, in terms of z the equation becomes

$$\frac{1}{1-n}\dot{z} + pz - q = 0 \tag{18}$$

and multiplying across by $1-n$ reduces the equation to a standard first order linear ordinary differential equation.

3. Find the general solution for

   (a) $\ddot{y} + 5\dot{y} + 6y = 0$

   (b) $\ddot{y} + 4\dot{y} - 21y = 0$

(c) $\ddot{y} + 11\dot{y} + 18y = 0$

In each example, what is the solution if $y(0) = 2$ and $\dot{y}(0) = -3$.

Solution: So the first one

$$\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) \tag{19}$$

so

$$y = C_1 e^{-2t} + C_2 e^{-3t} \tag{20}$$

and substituting in $y(0) = 2$ gives $C_1 + C_2 = 2$,

$$\dot{y} = -2C_1 e^{-2t} - 3C_2 e^{-3t} \tag{21}$$

hence $\dot{y}(0) = -3$ means $2C_1 + 3C_2 = 3$ and so $C_1 = 3$ and $C_2 = -1$ so

$$y = 3e^{-2t} - e^{-3t} \tag{22}$$

Now, for the second

$$\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3) \tag{23}$$

so

$$y = C_1 e^{-7t} + C_2 e^{3t} \tag{24}$$

and substituting in $y(0) = 2$ gives $C_1 + C_2 = 2$,

$$\dot{y} = -7C_1 e^{-7t} + 3C_2 e^{3t} \tag{25}$$

hence $\dot{y}(0) = -3$ means $7C_1 - 3C_2 = 3$ and so $C_1 = 9/10$ and $C_2 = 11/10$ so

$$y = \frac{9}{10} e^{-7t} + \frac{11}{10} e^{3t} \tag{26}$$

Finally

$$\lambda^2 + 11\lambda + 18 = (\lambda + 2)(\lambda + 9) \tag{27}$$

so

$$y = C_1 e^{-2t} + C_2 e^{-9t} \tag{28}$$

and substituting in $y(0) = 2$ gives $C_1 + C_2 = 2$,

$$\dot{y} = -2C_1 e^{-2t} - 9C_2 e^{-3t} \tag{29}$$

hence $\dot{y}(0) = -3$ means $2C_1 + 9C_2 = 3$ and so $C_1 = 15/7$ and $C_2 = -1/7$ so

$$y = -\frac{1}{7} e^{-2t} + \frac{15}{7} e^{-3t} \tag{30}$$