MA22S3 Fourier work sheet, outline solutions for six to nine.[1]

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Questions: some reasonably hard questions for practising your Fourier analysis.

6. Calculate the Fourier transform of the on-off pulse

$$f(t) = \begin{cases} -A & -d < t < 0 \\ A & 0 < t < d \\ 0 & |t| > T \end{cases} \quad (1)$$

Solution: So

$$2\pi\widetilde{f(k)} = \int_{-\infty}^{\infty} f(t)e^{-ikt}dt = -A\int_{-d}^{0} e^{-ikt}dt + A\int_{0}^{d} e^{-ikt}dt \quad (2)$$

Then, doing the integral,

$$\begin{aligned} 2\pi\widetilde{f(k)} &= \frac{A}{ik}\, e^{-ikt}\Big]_{-d}^{0} - \frac{A}{ik}\, e^{-ikt}\Big]_{0}^{d} \\ &= \frac{A}{ik}(1 - e^{ikd}) - \frac{A}{ik}(e^{-ikd} - 1) \\ &= \frac{2A}{ik}(1 - \cos kd) \end{aligned} \quad (3)$$

7. Define the Fourier cosine transform as $F_c(x) = \int_0^\infty f(t)\cos xt dt$ and show the function $f(t) = \cos{(at)}$ for $0 \geq t \geq a$ and zero otherwise, has

$$F_c(x) = \frac{1}{2}\left[\frac{\sin{(1+x)a}}{1+x} + \frac{\sin{(1-x)a}}{1-x}\right] \quad (4)$$

Solution: This is an integration

$$\begin{aligned} F_c(x) &= \int_0^\infty f(t)\cos xt dt \\ &= \int_0^a \cos at \cos xt dt = \frac{1}{2}\int_0^a (\cos{(a+x)t} + \cos{(a-x)t})\, dt \end{aligned} \quad (5)$$

where we have used a trignometric identity. Now, actually doing the integral, this gives

$$\begin{aligned} F_c(x) &= \frac{1}{2}\left(\frac{1}{a+x}\sin{(a+x)t} + \frac{1}{a-x}\sin{(a-x)t}\right)_0^a \\ &= \frac{1}{2}\left[\frac{\sin{(a+x)a}}{a+x} + \frac{\sin{(a-x)a}}{a-x}\right] \end{aligned} \quad (6)$$

which is different to the answer given in the question.

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8. The Fourier sine transform is $F_s(x) = \int_0^\infty f(t) \sin xt dt$; what are the Fourier sine and cosine transforms of

$$f(t) = \begin{cases} 1 & 0 \le t \le a \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Solution: So, again,

$$\begin{aligned} F_c(x) &= \int_0^\infty f(t) \cos xt dt = \int_0^a \cos xt dt = \frac{1}{x} \sin xt]_0^a = \frac{\sin xa}{x} \\ F_s(x) &= \int_0^\infty f(t) \sin xt dt = \int_0^a \sin xt dt = -\frac{1}{x} \cos xt]_0^a = \frac{1 - \cos xa}{x} \end{aligned} \tag{8}$$

9. If $\widetilde{f(k)}$ is the Fourier transform of $f(t)$ what is the Fourier transform of $f(t-a)$ where a is a constant.

   Solution: So just do it, let $f_a(t) = f(t-a)$ so $\widetilde{f_a(k)}$ is the Fourier transform of $f(t-a)$

$$2\pi \widetilde{f_a(k)} = \int_{-\infty}^{\infty} f(t-a) e^{-ikt} dt \tag{9}$$

then let $t' = t - a$ so

$$2\pi \widetilde{f_a(k)} = \int_{-\infty}^{\infty} f(t') e^{-ik(t'+a)} dt' = e^{-ika} \int_{-\infty}^{\infty} f(t') e^{-ikt'} dt' = 2\pi e^{-ika} \widetilde{f(k)} \tag{10}$$

so the effect of the shift is multiplication by a complex exponential.