MA22S3 Fourier work sheet, outline solutions for six to nine.[1]

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Questions: some reasonably hard questions for practising your Fourier analysis.

6. Calculate the Fourier transform of the on-off pulse

$$f(t)=\left\{\begin{array}{ll}-A & -d<t<0\\ A & 0<t<d\\ 0 & |t|>T\end{array}\right. \tag{1}$$

Solution: So

$$2\pi\widetilde{f(k)}=\int_{-\infty}^{\infty}f(t)e^{-ikt}dt=-A\int_{-d}^{0}e^{-ikt}dt+A\int_0^d e^{-ikt}dt \tag{2}$$

Then, doing the integral,

$$\begin{aligned}2\pi\widetilde{f(k)} &= \frac{A}{ik}\,e^{-ikt}\big]_{-d}^{0}-\frac{A}{ik}\,e^{-ikt}\big]_0^d\\ &= \frac{A}{ik}(1-e^{ikd})-\frac{A}{ik}(e^{-ikd}-1)\\ &= \frac{2A}{ik}(1-\cos kd)\end{aligned} \tag{3}$$

7. Define the Fourier cosine transform as $F_c(x)=\int_0^\infty f(t)\cos xtdt$ and show the function $f(t)=\cos{(at)}$ for $0\geq t\geq a$ and zero otherwise, has

$$F_c(x)=\frac{1}{2}\left[\frac{\sin{(1+x)a}}{1+x}+\frac{\sin{(1-x)a}}{1-x}\right] \tag{4}$$

Solution: This is an integration

$$\begin{aligned}F_c(x) &= \int_0^\infty f(t)\cos xtdt\\ &= \int_0^a \cos at\cos xtdt=\frac{1}{2}\int_0^a\left(\cos{(a+x)t}+\cos{(a-x)t}\right)dt\end{aligned} \tag{5}$$

where we have used a trignometric identity. Now, actually doing the integral, this gives

$$\begin{aligned}F_c(x) &= \frac{1}{2}\left(\frac{1}{a+x}\sin{(a+x)t}+\frac{1}{a-x}\sin{(a-x)t}\right)_0^a\\ &= \frac{1}{2}\left[\frac{\sin{(a+x)a}}{a+x}+\frac{\sin{(a-x)a}}{a-x}\right]\end{aligned} \tag{6}$$

which is different to the answer given in the question.

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8. The Fourier sine transform is $F_s(x)=\int_0^\infty f(t)\sin xtdt$; what are the Fourier sine and cosine transforms of

$$f(t)=\left\{\begin{array}{ll}1 & 0\leq t\leq a\\ 0 & \text{otherwise}\end{array}\right. \tag{7}$$

Solution: So, again,

$$\begin{aligned}F_c(x) &= \int_0^\infty f(t)\cos xtdt=\int_0^a\cos xtdt=\frac{1}{x}\sin xt]_0^a=\frac{\sin xa}{x}\\ F_s(x) &= \int_0^\infty f(t)\sin xtdt=\int_0^a\sin xtdt=-\frac{1}{x}\cos xt]_0^a=\frac{1-\cos xa}{x}\end{aligned} \tag{8}$$

9. If $\widetilde{f(k)}$ is the Fourier transform of $f(t)$ what is the Fourier transform of $f(t-a)$ where a is a constant.

Solution: So just do it, let $f_a(t)=f(t-a)$ so $\widetilde{f_a(k)}$ is the Fourier transform of $f(t-a)$

$$2\pi\widetilde{f_a(k)}=\int_{-\infty}^{\infty}f(t-a)e^{-ikt}dt \tag{9}$$

then let $t'=t-a$ so

$$2\pi\widetilde{f_a(k)}=\int_{-\infty}^{\infty}f(t')e^{-ik(t'+a)}dt'=e^{-ika}\int_{-\infty}^{\infty}f(t')e^{-ikt'}dt'=2\pi e^{-ika}\widetilde{f(k)} \tag{10}$$

so the effect of the shift is multiplication by a complex exponential.