MA22S3 Fourier work sheet, outline solutions, questions 4 and 5.[1]

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Questions: some reasonably hard questions for practising your Fourier analysis.

4. Calculate the complex Fourier series of the square pulse function, $f(t+2T)=f(t)$ and

$$f(t)=\begin{cases}0 & -T<t<-d\\ A & -d<t<d\\ 0 & d<t<T\end{cases} \tag{1}$$

Solution: So the period is $2T$ so, for $n!=0$

$$\begin{aligned} c_n &= \frac{1}{2T}\int_{-T}^{T} f(t)e^{-in\pi t/T}dt=\frac{A}{2T}\int_{-d}^{d}e^{-in\pi t/T}dt\\ &= \frac{A}{2ni\pi}\left(e^{in\pi d/T}-e^{-in\pi d/T}\right)=\frac{A}{n\pi}\sin\frac{n\pi d}{T} \end{aligned} \tag{2}$$

For $n=0$ we have $c_0=Ad/T$.

5. Derive the formula for c_n directly from the corresponding formulae for a_n and b_n.

Solution: So, for simplicity lets assume the period is 2π, the general case is not much different but requires more typing.

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt \tag{3}$$

Now, expand the two trignometric functions in terms of exponentials

$$f(t)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\left(e^{int}+e^{-int}\right)-ib_n\left(e^{int}-e^{-int}\right)\right] \tag{4}$$

so

$$f(t)=\frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\left[(a_n-ib_n)e^{int}+(a_n+ib_n)e^{-int}\right] \tag{5}$$

Matching up, this means $c_0=a_0/2$ and $2c_{\pm n}=a_n\mp ib_n$. Hence

$$c_n=\frac{a_n-ib_n}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\left(\cos nt-i\sin nt\right)dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}dt \tag{6}$$

as required, you can check the a_0 works too.

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