

MA22S3 Fourier work sheet, outline solutions, first three questions.¹

15 December 2010

Questions: some reasonably hard questions for practising your Fourier analysis, with some reasonably sketchy solutions, email me if you are still stuck.

1. What is the Fourier series for $f(t) = t^2$ for $t \in (-\pi, \pi)$ and $f(t + 2\pi) = f(t)$?

Solution: Lets do the trigonometric series; since $f(t)$ is even, we need only look at the a_n s. First a_0

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t) dt = \frac{2}{\pi} \int_0^{\pi} t^2 dt = \frac{2}{3} \pi^2 \quad (1)$$

where I have used to the evenness to go

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{2}{\pi} \int_0^{\pi} f(t) dt \quad (2)$$

Now, the a_n is more annoying since it will require two integrations by parts, so for $n > 0$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(t) \cos ntdt = \frac{2}{\pi} \int_0^{\pi} t^2 \cos ntdt \\ &= \frac{2}{\pi} \left[\frac{t^2}{n} \sin nt + \frac{2t}{n^2} \cos nt - \frac{2}{n^3} \sin nt \right]_0^{\pi} \\ &= \frac{2}{\pi} \left(\frac{2\pi}{n^2} \cos n\pi \right) = \frac{4}{n^2} (-1)^n \end{aligned} \quad (3)$$

Hence

$$f(t) = \frac{1}{3} \pi^2 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt \quad (4)$$

2. A periodic function $f(t)$ with period two is defined by

$$f(t) = \begin{cases} 3t & 0 < t < 1 \\ 3 & 1 < t < 2 \end{cases} \quad (5)$$

Sketch the function and calculate its Fourier series

Solution: So if you sketch the function it is a diagonal line from $(0, 0)$ to $(1, 3)$, then a horizontal line from $(1, 3)$ to $(2, 3)$; after that is repeat periodically. Sadly the whole thing is a mess, with lots of integrals to do, the main point is to remember that the

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period is $L = 2$; also, it is easier to take the integral from zero to two, rather than -1 to one, it doesn't make any difference to the result because of the periodicity.

$$a_0 = \int_0^2 f(t)dt = \int_0^1 3t dt + \int_1^2 3 dt = \frac{9}{2} \quad (6)$$

and

$$a_n = \int f(t) \cos n\pi t dt = \int_0^1 3t \cos n\pi t dt + \int_1^2 3 \cos n\pi t dt \quad (7)$$

doing these integrals, by parts in the case of the first one, we get $a_n = 0$ for n even and

$$a_n = -\frac{6}{(n\pi)^2} \quad (8)$$

for n odd. For b_n it is the n -squared term that disappears when you put in the integration limit and you get

$$b_n = -\frac{3}{n\pi} \quad (9)$$

3. Using the complex Fourier series of $f(t) = 2t/T$ on $0 < t < T$ and $f(t + 2T) = f(t)$ and Parseval's theorem, show

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (10)$$

Solution: This isn't that different from one of the problem sheet questions, the key point is that the period is $2T$ and, if $n \neq 0$

$$c_n = \frac{1}{2T} \int_0^{2T} f(t) e^{-in\pi t/T} dt = \frac{1}{2T} \int_0^{2T} \frac{2t}{T} e^{-in\pi t/T} dt = \frac{2i}{\pi n} \quad (11)$$

where the important point is that $e^{-2\pi i n} = 1$. If $n = 0$ the integral gives two. Hence

$$f(t) = 2 + \frac{2i}{\pi} \sum_{n \neq 0} \frac{1}{n} e^{in\pi t/T} \quad (12)$$

Next, we apply Parseval's theorem

$$\frac{1}{L} \int_0^L |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \quad (13)$$

giving, in this case

$$\frac{16}{3} = 4 + \sum_{n \neq 0} \frac{4}{n^2 \pi^2} \quad (14)$$

which gives the answer provided you note that $n^2 = (-n)^2$ so

$$\sum_{n \neq 0} \frac{1}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (15)$$