MA22S3 Fourier work sheet, outline solutions, first three questions.[1]

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Questions: some reasonably hard questions for practising your Fourier analysis, with some reasonably sketchy solutions, email me if you are still stuck.

1. What is the Fourier series for $f(t) = t^2$ for $t \in (-\pi, \pi)$ and $f(t + 2\pi) = f(t)$?

   Solution: Lets do the trignometric series; since $f(t)$ is even, we need only look at the a_ns. First a_0
   $$a_0 = \frac{2}{\pi}\int_0^\pi f(t)dt = \frac{2}{\pi}\int_0^\pi t^2 dt = \frac{2}{3}\pi^2 \tag{1}$$
   where I have used to the evenness to go
   $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{2}{\pi}\int_0^\pi f(t)dt \tag{2}$$
   Now, the a_n is more annoying since it will require two integrations by parts, so for $n > 0$
   $$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi f(t)\cos nt dt = \frac{2}{\pi}\int_0^\pi t^2 \cos nt dt \\ &= \frac{2}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_0^\pi \\ &= \frac{2}{\pi}\left(\frac{2\pi}{n^2}\cos n\pi\right) = \frac{4}{n^2}(-1)^n \end{aligned} \tag{3}$$
   Hence
   $$f(t) = \frac{1}{3}\pi^2 + 4\sum_{n=1}^\infty \frac{(-1)^n}{n^2}\cos nt \tag{4}$$

2. A periodic function $f(t)$ with period two is defined by
   $$f(t) = \begin{cases} 3t & 0 < t < 1 \\ 3 & 1 < t < 2 \end{cases} \tag{5}$$
   Sketch the function and calculate its Fourier series

   Solution: So if you sketch the function it is a diagonal line from $(0, 0)$ to (1,3), then a horizonal line from $(1, 3)$ to $(2, 3)$; after that is repeat periodically. Sadly the whole thing is a mess, with lots of integrals to do, the main point is to remember that the period is $L = 2$; also, it is easier to take the integral from zero to two, rather than -1 to one, it doesn't make any difference to the result because of the periodicity.
   $$a_0 = \int_0^2 f(t)dt = \int_0^1 3tdt + \int_1^2 3dt = \frac{9}{2} \tag{6}$$
   and
   $$a_n = \int f(t)\cos n\pi t dt = \int_0^1 3t\cos n\pi t dt + \int_1^2 3\cos n\pi t dt \tag{7}$$
   doing these integrals, by parts in the case of the first one, we get $a_n = 0$ for n even and
   $$a_n = -\frac{6}{(n\pi)^2} \tag{8}$$
   for n odd. For b_n it is the n-squared term that disappears when you put in the integration limit and you get
   $$b_n = -\frac{3}{n\pi} \tag{9}$$

3. Using the complex Fourier series of $f(t) = 2t/T$ on $0 < t < T$ and $f(t + 2T) = f(t)$ and Parseval's theorem, show
   $$\frac{\pi^2}{6} = \sum_{n=1}^\infty \frac{1}{n^2} \tag{10}$$
   Solution: This isn't that different from one of the problem sheet questions, the key point is that the period is $2T$ and, if $n! = 0$
   $$c_n = \frac{1}{2T}\int_0^{2T} f(t)e^{-in\pi t/T}dt = \frac{1}{2T}\int_0^{2T} \frac{2t}{T}e^{-in\pi t/T}dt = \frac{2i}{\pi n} \tag{11}$$
   where the important ppint is that $e^{-2\pi i n} = 1$. If $n = 0$ the integral gives two. Hence
   $$f(t) = 2 + \frac{2i}{\pi}\sum_{n!=0} \frac{1}{n}e^{in\pi t/T} \tag{12}$$
   Next, we apply Parceval's theorem
   $$\frac{1}{L}\int_0^L |f(t)|^2 dt = \sum_{n=-\infty}^\infty |c_n|^2 \tag{13}$$
   giving, in this case
   $$\frac{16}{3} = 4 + \sum_{n!=0} \frac{4}{n^2\pi^2} \tag{14}$$
   which gives the answer provided you note that $n^2 = (-n)^2$ so
   $$\sum_{n!=0} \frac{1}{n^2} = 2\sum_{n=1}^\infty \frac{1}{n^2} \tag{15}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`