MA22S3 Outline Solutions for Tutorial Sheet 7.[1]

9 December 2010

Questions

1. (2) Obtain the general solution to

$$\ddot{y} + 8\dot{y} + 16y = 0 \quad (1)$$

Solution: So, first of all write down the auxiliary equation

$$\lambda^2 + 8\lambda + 16 = 0 \quad (2)$$

which factorizes as

$$(\lambda + 4)^2 = 0 \quad (3)$$

hence there is only one solution $\lambda = -4$ and the solution is

$$y = C_1 e^{-4t} + C_2 t e^{-4t} \quad (4)$$

2. (2) Obtain the general solution to

$$\ddot{y} + 16y = 0 \quad (5)$$

Solution: Again, first find the auxiliary equation

$$\lambda^2 + 16 = 0 \quad (6)$$

so

$$\lambda = \pm 4i \quad (7)$$

and hence the solution is

$$y = C_1 e^{4it} + C_2 e^{-4it} \quad (8)$$

This is real provided $C_1^* = C_2$ so let $C_1 = (A - iB)/2$ and hence $C_2 = (A + iB)/2$ where A and B are real. Putting the minus in the C_1 and having the half is for niceness, it gives a slightly neater answer, but isn't important. Now

$$\begin{aligned} y &= \frac{1}{2}(A - iB)(\cos 4t + i\sin 4t) + \frac{1}{2}(A + iB)(\cos 4t - \sin 4t) \\ &= A\cos 4t + B\sin 4t \end{aligned} \quad (9)$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3

3. (2) Obtain the solution to

$$\ddot{y} + \dot{y} - 2y = e^{5t} \tag{10}$$

with $y(0) = \dot{y}(0) = 0$.

Solution: So first of all we have to find the complementary function, the solution to

$$\ddot{y} + \dot{y} - 2y = 0 \tag{11}$$

We do this by first calculating the auxiliary equation

$$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0 \tag{12}$$

so

$$y_c = C_1 e^{-2t} + C_2 e^t \tag{13}$$

Next, we need a particular solution, since neither of the roots of the auxiliary equation is five this is straight forward, we just substitute

$$y = Ce^{5t} \tag{14}$$

to get

$$25Ce^{5t} + 5Ce^{5t} - 2Ce^{5t} = e^{5t} \tag{15}$$

and hence $28C = 1$, so the general solution is

$$y = C_1 e^{-2t} + C_2 e^t + \frac{1}{28} e^{5t} \tag{16}$$

Now, we want $y(0) = 0$ which implies

$$C_1 + C_2 + \frac{1}{28} = 0 \tag{17}$$

and $\dot{y}(0) = 0$ which implies

$$-2C_1 + C_2 + \frac{5}{28} = 0 \tag{18}$$

Taking the bottom equation from the top one gives

$$3C_1 = \frac{4}{28} \tag{19}$$

or $C_1 = 4/84$ and hence $C_2 = -7/84$ giving

$$y = \frac{1}{21} e^{-2t} - \frac{7}{84} e^t + \frac{1}{28} e^{5t} \tag{20}$$

4. (2) Obtain the general solution to

$$\ddot{y} + 8\dot{y} + 16y = 4\cosh t \tag{21}$$

Solution: So the complementary function is know here, the homogeneous equation is the same as the equation in problem 1

$$y_c = C_1 e^{-4t} + C_2 t e^{-4t} \tag{22}$$

Now we need to solve the full equation, since

$$4\cosh t = 2e^t + 2e^{-t} \tag{23}$$

we will solve

$$\ddot{y} + 8\dot{y} + 16y = 2e^t \tag{24}$$

Since the unique solution to $\lambda^2 + 8\lambda + 16 = 0$ isn't one, we substitute in

$$y = Ce^t \tag{25}$$

to get

$$C + 8C + 16C = 2 \tag{26}$$

or $C = 2/25$; hence

$$y = \frac{2}{25}e^t \tag{27}$$

is a particular integral for the first term, now lets solve

$$\ddot{y} + 8\dot{y} + 16y = 2e^t \tag{28}$$

Again, the exponent on the right hand side doesn't match anything appearing in the complementary function so we can substitute

$$y = Ce^{-t} \tag{29}$$

giving

$$C - 8C + 16C = 2 \tag{30}$$

and hence $C = 2/9$ and so the second term has particular integral

$$y = \frac{2}{9}e^{-t} \tag{31}$$

Putting all this together we get the solution to the original equation as

$$y = C_1 e^{-4t} + C_2 t e^{-4t} + \frac{2}{25}e^t + \frac{2}{9}e^{-t} \tag{32}$$