MA22S3 Tutorial Sheet 6, outline solutions.[1][2]

2 December 2010

Questions

1. (2) Obtain the solution to $\dot{y} - 3y = e^{-t}$ with $y(0) = 1$; since an initial condition is chosen at $t = 0$, choose $a = 0$.

   Solution: To plug it into the equation, we have $p = -3$ so $I(t) = int_a^t p(t)dt = -3t + 3a = -3t$ if a is set at $a = 0$ hence, from the formula,

   $$y(t) = y(0)e^{3t} + e^{3t}\int_0^t e^{-\tau}e{-}3\tau d\tau = e^{3t} - \frac{1}{4}\left(e^{-t} - e^{3t}\right) = \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t} \tag{1}$$

   Alternatively, rewrite as

   $$e^{-3t}y' - 3ye^{-3t} = e^{-4t}$$

   or

   $$\left(e^{-3t}y\right)' = e^{-4t}$$

   and then integrate.

2. (2) Obtain a general solution to $(t+1)\dot{y} + y = (t+1)^2$

   Solution: $(t+1)\dot{y} + y = (t+1)^2$ can again be rewritten

   $$\frac{d}{dt}[(t+1)y] = t^2 + 2t + 1 \tag{2}$$

   so

   $$(t+1)y = \frac{1}{3}t^3 + t^2 + t + C \tag{3}$$

   or

   $$y = \frac{t^3 + 3t^2 + 3t + 1}{3(t+1)} + \frac{C}{t+1} = \frac{1}{3}(t+1)^2 + \frac{C}{t+1} \tag{4}$$

   with a redefinition of C to get the nice devision at the end, another way to do this would have been to change variables to $z = t + 1$ at the start.

   On the other hand dividing across by $1+t$ means we have to consider $p(t) = 1/(1+t)$; we can go down this route, it is just a question of putting all the constants together as C at the end.

   $$\int_a^t \frac{1}{1+\tau}d\tau = \ln(1+t) - \ln(1+a) \tag{5}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

[2] Including material from Chris Ford, to whom many thanks.

so lets choose $a = 0$ for convenience. Now $\exp \ln{(1+t)} = 1+t$ and, noting that $f = (1+t)$ since we divide across by $(1+t)$ to get the equation in to standard form

$$\int_0^t e^{I(\tau)} f(\tau) d\tau = \int_0^t (1+\tau)^2 d\tau = \frac{1}{3}(1+t)^3 - \frac{1}{3} \tag{6}$$

so substituting in to the equation

$$y(t) = \frac{y(0)}{1+t} + \frac{1}{3}(1+t)^2 - \frac{1}{3(1+t)} \tag{7}$$

and so we get the same answer if we define $C = y(0) - 1/3$.

3. (2) Obtain the general solution to

$$\ddot{y} + \dot{y} - 2y = 0 \tag{8}$$

Solution: The auxiliary equation is

$$\lambda^2 + \lambda - 2 = 0 \tag{9}$$

This is factorized to give

$$(\lambda - 1)(\lambda + 2) = 0 \tag{10}$$

so the solutions are $\lambda_1 = 1$ and $\lambda_2 = -2$, so the solution is

$$y = C_1 e^t + C_2 e^{-2t} \tag{11}$$

4. (2) Obtain the general solution to

$$\ddot{y} + 6\dot{y} + 8y = 0 \tag{12}$$

Solution: The auxiliary equation is

$$\lambda^2 + 6\lambda + 8 = 0 \tag{13}$$

This is factorized to give

$$(\lambda + 4)(\lambda + 2) = 0 \tag{14}$$

so the solutions are $\lambda_1 = -4$ and $\lambda_2 = -2$, so the solution is

$$y = C_1 e^{-4t} + C_2 e^{-2t} \tag{15}$$