MA22S3 Tutorial Sheet 5, outline solutions.[1,2]

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Questions

1. (2) Revise Fourier series and Parceval's Theorem. Consider the Fourier expansion of $f(t) = t, \quad -\pi < t < \pi$ with $f(t + 2\pi) = f(t)$ and use the result to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution: So first of all

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} t dt = 0 \tag{1}$$

since the integrand is odd. The other coefficients are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} t e^{-ikt} dt \tag{2}$$

Doing this by parts and using

$$e^{in\pi} = e^{-in\pi} = (-1)^n \tag{3}$$

gives

$$c_n = \frac{i(-1)^n}{n} \tag{4}$$

Now, we apply Parceval's theorem, since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3} \tag{5}$$

we get

$$\sum_{n=-\infty}^{-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \tag{6}$$

Since there is an n^2 in the summand, the result is the same for the sum over negative and over positive numbers, so

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{7}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/MA22S3
[2]Including material from Chris Ford, to whom many thanks.

2. (2) Show that the Fourier transform of an even function is even, that is, if $f(-t) = f(t)$ then $\widetilde{f(-k)} = \widetilde{f(k)}$.

   Solution: Assume that f is even, that is $f(-t) = f(t)$, then

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\ e^{ikt} f(t).$$

   make the change of variables $s = -t$:

$$\tilde{f}(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\ e^{-iks} f(-s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\ e^{-iks} f(s) = \tilde{f}(k).$$

3. (2) Use integration by parts to show that

$$\int_{-infty}^{\infty} f(t)\frac{d}{dt}\delta(t)dt = -\dot{f}(0) \tag{8}$$

   where we are using the notation

$$\frac{d}{dt}f(t) = f(\dot{t}) \tag{9}$$

   Solution: So in

$$\int_{-infty}^{\infty} f(t)\frac{d}{dt}\delta(t)dt \tag{10}$$

   let $u = f(t)$ and $dv = \delta(\dot{t})dt$ so, by parts, we get

$$\int_{-infty}^{\infty} f(t)\frac{d}{dt}\delta(t)dt = f(t)\delta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(\dot{t})\delta(t)dt \tag{11}$$

   The first term on the right hand side is zero because the delta function is zero when $t \neq 0$, we can evaluate the second term using the delta function, givng the answer.

4. (2) By substituting $y = Ae^{rt}$ and solving for r obtain the solution to $\dot{y} - 3y = 0$, if $y(0) = 1$ calculate the value of A.

   Solution: If $y = Ae^{rt}$ then $\dot{y} = Are^{rt}$ so the equation becomes

$$Are^{rt} - 3Ae^{rt} = 0 \tag{12}$$

   so cancelling the Ae^{rt} we get $r = 3$. Now if $y(t) = Ae^{3t}$ then $y(0) = A$, hence if $y(0) = 1$, $A = 1$ and we get

$$y(t) = e^{3t} \tag{13}$$