MA22S3 Tutorial Sheet 4, outline solutions.[1][2]

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Questions

1. (3) Compute the Fourier transform of $f(t) = te^{-t}$ for $t > 0$ and zero otherwise.

   Solution: So we need the integral

$$\widetilde{f(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, f(t)e^{-ikt} = \frac{1}{2\pi}\int_{0}^{\infty} dtte^{-(ik+1)t} \tag{1}$$

   Integrating by parts

$$2\pi\widetilde{f(k)} = \int_0^\infty dtte^{-(ik+1)t} = -\left.\frac{1}{ik+1}te^{-(ik+1)t}\right]_0^\infty + \frac{1}{ik+1}\int_0^\infty dte^{-(ik+1)t} \tag{2}$$

   Now the first term gives zero and so

$$2\pi\widetilde{f(k)} = \frac{1}{(1+ik)^2} \tag{3}$$

2. (2) Using integration by parts or otherwise, write the Fourier transform of df/dt in terms of the Fourier transform of f.

   Solution: So

$$\mathcal{F}\left(\frac{df}{dt}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{df}{dt}e^{-ikt}dt = \frac{1}{2\pi}\left. fe^{-ikt}\right]_{-\infty}^{\infty} + ik\frac{1}{2\pi}\int_{-\infty}^{\infty} fe^{-ikt}dt = ik\mathcal{F}(f) \tag{4}$$

   using the vanishing of f at plus and minus infinity.

3. (3) Do the following integrals

   (a)

$$\int_{-\infty}^{\infty}\delta(t)(t^2+3t+5)dt \tag{5}$$

   (b)

$$\int_{-\infty}^{\infty}\delta(t-\pi/4)\sin t dt \tag{6}$$

   (c)

$$\int_{-\infty}^{\infty}\delta(t-1)\ln t dt \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

[2] Including material from Chris Ford, to whom many thanks.

Solution: The argument of the delta function is zero for $t = 0$ and that's included in the range of integration so we just evaluate the integrand at $t = 0$.

$$\int_{-\infty}^{\infty} \delta(t)(t^2 + 3t + 5)dt = 5 \tag{8}$$

Next

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt \tag{9}$$

Here the arguement of the delta function is zero when $t = \pi/4$ and this is in the range of integration, so

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{10}$$

Finally

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt \tag{11}$$

Here, we substitute $t = 1$ into the integrand to give

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt = \ln 1 = 0 \tag{12}$$