

MA22S3 Tutorial Sheet 4, outline solutions.¹²

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Questions

1. (3) Compute the Fourier transform of $f(t) = te^{-t}$ for $t > 0$ and zero otherwise.

Solution: So we need the integral

$$\widetilde{f(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f(t) e^{-ikt} = \frac{1}{2\pi} \int_0^{\infty} dt t e^{-(ik+1)t} \quad (1)$$

Integrating by parts

$$2\pi \widetilde{f(k)} = \int_0^{\infty} dt t e^{-(ik+1)t} = -\left[\frac{1}{ik+1} t e^{-(ik+1)t} \right]_0^{\infty} + \frac{1}{ik+1} \int_0^{\infty} dt e^{-(ik+1)t} \quad (2)$$

Now the first term gives zero and so

$$2\pi \widetilde{f(k)} = \frac{1}{(1+ik)^2} \quad (3)$$

2. (2) Using integration by parts or otherwise, write the Fourier transform of df/dt in terms of the Fourier transform of f .

Solution: So

$$\mathcal{F}\left(\frac{df}{dt}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{df}{dt} e^{-ikt} dt = \frac{1}{2\pi} f e^{-ikt} \Big|_{-\infty}^{\infty} + ik \frac{1}{2\pi} \int_{-\infty}^{\infty} f e^{-ikt} dt = ik \mathcal{F}(f) \quad (4)$$

using the vanishing of f at plus and minus infinity.

3. (3) Do the following integrals

(a)

$$\int_{-\infty}^{\infty} \delta(t)(t^2 + 3t + 5) dt \quad (5)$$

(b)

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt \quad (6)$$

(c)

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt \quad (7)$$

Solution: The argument of the delta function is zero for $t = 0$ and that's included in the range of integration so we just evaluate the integrand at $t = 0$.

$$\int_{-\infty}^{\infty} \delta(t)(t^2 + 3t + 5) dt = 5 \quad (8)$$

Next

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt \quad (9)$$

Here the argument of the delta function is zero when $t = \pi/4$ and this is in the range of integration, so

$$\int_{-\infty}^{\infty} \delta(t - \pi/4) \sin t dt = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad (10)$$

Finally

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt \quad (11)$$

Here, we substitute $t = 1$ into the integrand to give

$$\int_{-\infty}^{\infty} \delta(t - 1) \ln t dt = \ln 1 = 0 \quad (12)$$

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