MA22S3 Tutorial Sheet 3, outline solutions.[1][2]

2 November 2010

Questions

1. (4) Compute the Fourier transform of $f(t) = e^{-a|t|}$ where a is a positive constant.

   Solution: The idea here is to plug $f(t)$ directly in to the equation for $\tilde{f}(k)$ and then use $|t| = t$ for $t > 0$ and $|t| = -t$ for $t < 0$.

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-ikt}\; f(t) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-ikt}\; e^{-a|t|} = \frac{1}{2\pi}\left[\int_{0}^{\infty} dt\; e^{-ikt-at} + \int_{-\infty}^{0} dt\; e^{-ikt+at}\right] \\
&= \frac{1}{2\pi}\left[-\frac{e^{-t(a+ik)}}{a+ik}\bigg|_0^{\infty} - \frac{e^{t(a-ik)}}{a-ik}\bigg|_{-\infty}^{0}\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \frac{1}{\pi}\frac{a}{a^2+k^2}.
\end{aligned}$$

   f can be represented as a Fourier integral

$$f(t) = \int_{-\infty}^{\infty} dk\; e^{ikt}\; \tilde{f}(k) = \frac{a}{\pi}\int_{-\infty}^{\infty} dk\; \frac{e^{ikt}}{a^2+k^2}.$$

2. (4) Express the following function as a Fourier integral:

$$f(x) = \begin{cases} \cos t & |t| < \frac{\pi}{2} \\ 0 & |t| > \frac{\pi}{2} \end{cases}$$

   One way to do the required integral is to split the cosine into exponentials.

   Solution: (a) Writing f as a Fourier integral $f(t) = \int_{-\infty}^{\infty} e^{ikt}\; \tilde{f}(k)\; dk$. We require the Fourier transform, the important thing is to note that the function is zero outside of $(-\pi/2, \pi/2)$ so the integral outside of that range doesn't contribute to the answer

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\; e^{-ikt}\; f(t) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt\; e^{-ikt}\; \frac{e^{it}+e^{-it}}{2} \\
&= \frac{1}{4\pi}\left(\frac{e^{i(1-k)t}}{i(1-k)} + \frac{e^{i(-1-k)t}}{i(-1-k)}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-ik\pi/2} + ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2} - ie^{ik\pi/2}}{i(-1-k)}\right]
\end{aligned}$$

[1]Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/MA22S3`

[2]Including material from Chris Ford, to whom many thanks.

$$= \frac{1}{4\pi} \, 2\cos\left(\frac{k\pi}{2}\right) \, \left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \frac{1}{\pi}\cos\left(\frac{k\pi}{2}\right) \, \frac{1}{1-k^2}.$$

Therefore

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk \; \cos\left(\frac{k\pi}{2}\right) \, \frac{e^{ikt}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$.