

# MA22S3 Tutorial Sheet 3, outline solutions.<sup>12</sup>

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## Questions

1. (4) Compute the Fourier transform of  $f(t) = e^{-a|t|}$  where  $a$  is a positive constant.

*Solution:* The idea here is to plug  $f(t)$  directly in to the equation for  $\tilde{f}(k)$  and then use  $|t| = t$  for  $t > 0$  and  $|t| = -t$  for  $t < 0$ .

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-ikt} f(t) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-ikt} e^{-a|t|} = \frac{1}{2\pi} \left[ \int_0^{\infty} dt e^{-ikt-at} + \int_{-\infty}^0 dt e^{-ikt+at} \right] \\ &= \frac{1}{2\pi} \left[ -\frac{e^{-t(a+ik)}}{a+ik} \Big|_0^{\infty} - \frac{e^{t(a-ik)}}{a-ik} \Big|_{-\infty}^0 \right] \\ &= \frac{1}{2\pi} \left[ \frac{1}{a+ik} + \frac{1}{a-ik} \right] = \frac{1}{\pi} \frac{a}{a^2 + k^2}.\end{aligned}$$

$f$  can be represented as a Fourier integral

$$f(t) = \int_{-\infty}^{\infty} dk e^{ikt} \tilde{f}(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk \frac{e^{ikt}}{a^2 + k^2}.$$

2. (4) Express the following function as a Fourier integral:

$$f(x) = \begin{cases} \cos t & |t| < \frac{\pi}{2} \\ 0 & |t| > \frac{\pi}{2} \end{cases}$$

One way to do the required integral is to split the cosine into exponentials.

*Solution:* (a) Writing  $f$  as a Fourier integral  $f(t) = \int_{-\infty}^{\infty} e^{ikt} \tilde{f}(k) dk$ . We require the Fourier transform, the important thing is to note that the function is zero outside of  $(-\pi/2, \pi/2)$  so the integral outside of that range doesn't contribute to the answer

$$\begin{aligned}\tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-ikt} f(t) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dt e^{-ikt} \frac{e^{it} + e^{-it}}{2} \\ &= \frac{1}{4\pi} \left( \frac{e^{i(1-k)t}}{i(1-k)} + \frac{e^{i(-1-k)t}}{i(-1-k)} \right) \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{4\pi} \left[ \frac{ie^{-ik\pi/2} + ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2} - ie^{ik\pi/2}}{i(-1-k)} \right]\end{aligned}$$

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<sup>2</sup>Including material from Chris Ford, to whom many thanks.

$$= \frac{1}{4\pi} 2 \cos\left(\frac{k\pi}{2}\right) \left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \frac{1}{\pi} \cos\left(\frac{k\pi}{2}\right) \frac{1}{1-k^2}.$$

Therefore

$$f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \cos\left(\frac{k\pi}{2}\right) \frac{e^{ikt}}{1-k^2}.$$

Remark:  $\tilde{f}(k)$  is well behaved at  $k = \pm 1$ .